FRESHMAN MATHEMATICS

MULLINS AND SMITH

FRESHMAN MATHEMATICS

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GINN AND COMPANY

BOSTON · NEW YORK · CHICAGO · LONDON
ATLANTA · DALLAS · COLUMBUS · SAN FRANCISCO

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PRINTED IN THE UNITED STATES OF AMERICA

943.4

The Athenæum Press
GINN AND COMPANY · PRO-
PRIETORS · BOSTON · U.S.A.

PREFACE

This book is written for the purpose of giving students who are beginning their college work, or who are taking advanced courses in secondary schools, a general view of the meaning of the mathematics that immediately follow the rudiments of algebra and geometry. The work presupposes only an elementary course in algebra such as is offered in every high school, a familiarity with the important basal propositions of plane geometry, and a knowledge of the simple rules or formulas of mensuration commonly given in arithmetic or in intuitive geometry. It is being generally recognized that such an all-round view of mathematics can best be introduced through the common uses of the science, leaving the rigorous treatment of the abstract theory to be taken up later by those who plan to specialize in the subject. It is upon this principle that the present book is planned; that is, the need for each topic is shown first and the work is then developed by means of applications such as the student is likely to meet in subsequent courses in science, economics, mathematics, or other subjects.

There is not so much uniformity of opinion, however, in regard to the details of the work of the freshman year. Some instructors prefer to arrange their own courses, using separate textbooks in college algebra, trigonometry, solid geometry, and sometimes analytic geometry and the calculus, but many wish a single textbook which gives, as this one seeks to give, a definite sequence of topics that shall serve as an introduction to college mathematics. It is the aim of the authors to provide in this book the material to meet this latter need; that is, to present the basis of a one-year course

which may be said to open the door of college mathematics to all by showing the meaning and the purpose of the science and its general usefulness in the various fields of intellectual activity.

Instead of beginning with the conventional formal review of algebra, the plan has been adopted of setting forth clearly the types of work in which the student is likely to need algebra in his subsequent study, and following this with a review that is limited strictly to the essentials of the subject. By being based upon the principle of subsequent usefulness, this review can be placed upon a somewhat higher plane than is possible with the conventional type. As the work proceeds, other uses of mathematics naturally present themselves, until the student finds himself in possession of a fair working knowledge of algebra, trigonometry, analytics, and the calculus such as he will need in beginning the study of the various sciences or in the introductory work in courses in distinctively college mathematics.

It is the hope of the authors that this book will thus meet the need in many colleges for a presentation of concrete mathematics that will stimulate students to continue their interest in a science which touches so many phases of human activity.

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CONTENTS

CHAPTER PAGE

I. ELEMENTARY ALGEBRA APPLIED

1. The Chief Need for Algebra 1
2. The Equation 9
3. Graphs 25
4. Generalized Exponents 45

II. THE BINOMIAL THEOREM AND SERIES

1. The Binomial Theorem 53
2. Arithmetic Series 57
3. Geometric Series 61

III. LOGARITHMS

1. General Principles 65
2. Computations with Logarithms 73

IV. TRIGONOMETRY

1. Trigonometric Functions 83
2. The Right Triangle 123
3. The Oblique Triangle 133
4. Problems from Mechanics and Physics 155
5. Analytic Trigonometry 161

V. ANALYTIC GEOMETRY

1. General Nature of the Work 163
2. Inclination and Slope 173
3. Straight-Line Loci 177
4. Circular Loci 195
5. The Parabola 203
6. The Ellipse 211
7. The Hyperbola 217
8. Miscellaneous Loci 223

CHAPTER PAGE

VI. THE CALCULUS

1. Maximum and Minimum Values 239
2. Variables 243
3. The Derivative 257
4. Tangents and Normals 277
5. Derivatives in Relation to Maxima and Minima 281
6. Differentials 287
7. Integration 293

VII. NUMERICAL EQUATIONS

1. Polynomials 317
2. Theory of Equations 323

VIII. PRACTICAL MENSURATION

1. Plane Figures 333
2. Solid Figures 339

TABLES

1. Powers and Roots 351
2. Logarithms of Numbers 357
3. Trigonometric Functions 360
4. Common Measures 382

INDEX 383

FRESHMAN MATHEMATICS

CHAPTER I

ELEMENTARY ALGEBRA APPLIED

1. The Chief Need for Algebra

Formulas. Algebra is, to a large extent, concerned with the study of rules expressed concisely by means of *formulas*. The formulas may be very simple ones like

$$A = \pi r^2,$$

which represents the area of a circle; they may involve several letters as in the case of the formula

$$L = 3\tfrac{1}{7}(R + r) + 2\,d,$$

which gives the approximate length of a belt that passes around two pulleys; they may involve a quadratic equation, as in the case of the gas-engine formula

$$P = 2\,c - 0.01\,c^2;$$

or, as is frequently the case in engineering and similar work, they may involve more advanced mathematics.

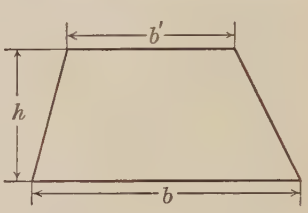
In every case we are primarily seeking to formulate in mathematical language the problem that we are considering, and thus to solve the problem. In so doing we may use a familiar formula, we may need to derive a new formula from one or more given ones, or we may state the problem in the language of an equation. Since solving an equation, however, is essentially like deriving one formula from another, algebra may with considerable propriety be called the science of the formula.

Using Formulas. In making use of formulas the three important points that we need to consider are as follows:

1. *The significance of the formula.*

In finding the area of a trapezoid by means of the formula

$$A = \tfrac{1}{2}\, h(b + b')$$



we need to know that it means that the area of the trapezoid is half the product of the height and the sum of the bases. Similarly, we need to know the technical meaning of any other formula of which we wish to make practical use.

For the purposes of algebra, however, this is not always essential. For example, we can easily solve for V the formula

$$T = \frac{WV^2}{32.2},$$

which is used in physics in connection with pulleys, without knowing its technical meaning, because we see that it is an equation which sets forth the relation between the three quantities that are represented by T, W, and V. That is, in such cases it is sufficient to know the general significance of the formula.

2. *Evaluating the formula.*

To evaluate the above formula for the trapezoid means to find the value of A for specified values of h, b, and b'. For example, if $h = 10$ ft., $b = 20$ ft., and $b' = 16$ ft., we have

$$A = \tfrac{1}{2}\, h(b + b') = \tfrac{1}{2} \times 10 \times (20 + 16) = 180,$$

which means that the area of the trapezoid is 180 sq. ft.

3. *Deriving one formula from another.*

In solving for V the pulley formula given above, we are simply deriving a formula for V from the one for T.

Exercises. Formulas

1. In the formula $A = bh$, which gives the area of a rectangle, A represents the area, b the base, and h the height. Write the rule expressed by the formula.

Write the rules expressed by the following formulas:

2. $A = \frac{1}{2}bh$ (triangle).

3. $C = 2\pi r$ (circle).

4. $C = \pi d$ (circle).

5. $V = e^3$ (cube).

6. $A = \pi r^2$ (circle).

7. $A = \frac{1}{4}\pi d^2$ (circle).

8. $V = \pi r^2 h$ (circular cylinder).

9. $V = \frac{1}{3}\pi r^2 h$ (circular cone).

Write formulas for the rules in Exs. 10–12:

10. The volume of a prism is the product of the area of the base and the height.

11. The volume of a sphere is one sixth the product of π and the cube of the diameter.

12. The square of the sum of two numbers is the sum of the squares of the numbers increased by twice their product.

13. Evaluate the formula $V = \frac{1}{3}\pi r^2 h$, which gives the volume of a circular cone, for $r = 6.3$ in. and $h = 9.5$ in.; for $r = 9.5$ in. and $h = 6.3$ in.

In such work take π as $\frac{22}{7}$ unless otherwise directed.

14. Evaluate the formula $V = \frac{1}{3}h(B + B' + \sqrt{BB'})$, which gives the volume of a conic frustum with the height h and with bases having the areas B and B' respectively, for $B = 18$ sq. in., $B' = 8$ sq. in., and $h = 11.5$ in.

15. Evaluate the formula $r = \sqrt{\frac{V}{\pi h}}$, which gives the radius of a circular cylinder with the volume V and the height h, for $V = 110$ cu. in. and $h = 7$ in.

In finding square roots, the tables on pages 352–356 may be used whenever they can be conveniently applied.

Factoring in Formulas. If we wish to evaluate the formula

$$A = \pi r^2 - \pi r'^2,$$

which gives the metallic cross-section area of a water pipe, for $r = 6.8$ in. and $r' = 5.8$ in., we might substitute these values in the formula and proceed as follows:

$$\begin{aligned} A = \pi r^2 - \pi r'^2 &= \tfrac{22}{7} \times 6.8^2 - \tfrac{22}{7} \times 5.8^2 \\ &= \tfrac{22}{7} \times 46.24 - \tfrac{22}{7} \times 33.64 \\ &= \tfrac{1}{7} \times 1017.28 - \tfrac{1}{7} \times 740.08 \\ &= 145.33 - 105.73 \\ &= 39.60. \end{aligned}$$

That is, to the nearest 0.1 sq. in., the cross-section area is 39.6 sq. in. Such a cumbersome method is easily avoided by first factoring the right-hand side as follows:

$$A = \pi r^2 - \pi r'^2 = \pi(r^2 - r'^2) = \pi(r + r')(r - r').$$

Then substituting the given values, we have

$$A = \tfrac{22}{7}(6.8 + 5.8)(6.8 - 5.8) = \tfrac{22}{7} \times 12.6 \times 1 = 39.6.$$

The advantages of knowing how to factor such an expression are readily apparent from this example. We may therefore say that, in general,

To save work, factor whenever possible.

For example, the formula for the area of the total surface of a circular cylinder,

$$T = 2\,\pi rh + 2\,\pi r^2,$$

is much simpler for computation when it is written in the form

$$T = 2\,\pi r(h + r).$$

Furthermore, the factored form in a case like the trapezoid formula

$$A = \tfrac{1}{2}\,h(b + b')$$

is more convenient for deriving a formula for h in terms of A, b, and b'. In fact, a very important purpose of factoring is to render formulas more simple for computation or manipulation.

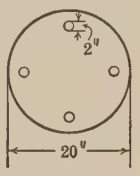
Exercises. Factoring in Formulas

1. Evaluate the formula $S = a^2 + 2ab + b^2$ for $a = 9.73$ and $b = 6.27$, first changing the right-hand side to a form more convenient for computation.

2. Evaluate the formula $D = x^2 - 2xy + y^2$ for $x = 7.3$ and $y = 4.3$, without first writing it in a different form. Then write the formula in a more convenient form for computation and evaluate.

3. Write the formula $P = m^2 - n^2$ in a more convenient form and then evaluate it for $m = 9.27$ and $n = 8.47$.

4. Write a formula in factored form for the area of a metal plate 20 in. in diameter pierced by four holes, each 2 in. in diameter, and then find the area of the plate.

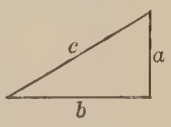


In practical work, 1 ft. 6 in. is usually written as 1′6″.

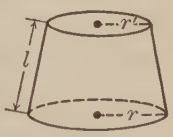
5. Find the value of $ab + ac$ when $a = 17.3$, $b = 7.6$, and $c = 2.4$, first taking the expression as given and then after putting it in a more convenient form.

6. Find the value of $64x^2 - 9y^2$ when $x = 2.2$ and $y = 1.3$.

7. The formula for the length of the side a of this right triangle is $a = \sqrt{c^2 - b^2}$. Find in the simplest way the value of a when $c = 9.7$ in. and $b = 5.3$ in.



8. The area of the curve surface of a right circular conic frustum is given by the formula $A = \pi lr + \pi lr'$, where l is the slant height of the frustum and r and r' are the radii of the lower and upper bases respectively. Find the area of the curve surface of such a frustum in which $l = 11.2$ in., $r = 5.4$ in., and $r' = 3.8$ in., first changing the right-hand side to a more convenient form.



Important Cases of Factoring. Parts of the usual treatment of factoring in elementary algebra do not have much value in connection with the work we are to consider. The five following cases, however, are of importance:

$$ax + ay = a(x + y) \tag{1}$$

$$a^2 + 2ab + b^2 = (a + b)^2 \tag{2}$$

$$a^2 - 2ab + b^2 = (a - b)^2 \tag{3}$$

$$a^2 - b^2 = (a + b)(a - b) \tag{4}$$

$$x^2 + (a + b)x + ab = (x + a)(x + b) \tag{5}$$

Case 1 includes all cases of removing a monomial factor, and represents a very important type.

It, of course, includes cases like $5x^3 + 10x^2 - 25x = 5x(x^2 + 2x - 5)$, and similar ones. This case is often valuable in saving work in evaluating expressions, as may be seen by evaluating $3ax^2 + ay - 2az^2$ for $a = 1.1$, $x = \frac{2}{3}$, $y = \frac{1}{3}$, and $z = \frac{1}{4}$.

Cases 2 and 3, which differ only with respect to signs, include all squares of binomial expressions.

For example, it includes cases like $25m^2 + 40m + 16 = (5m + 4)^2$ and $9m^4 - 12m^2 + 4 = (3m^2 - 2)^2$. Although we do not often need to evaluate such expressions, we frequently need to factor them.

Case 4 is, next to the first one, the type most frequently needed. We have already seen several uses of it, and we shall find others as we go on.

It includes cases like $a^2 + 2ab + b^2 - c^2 = (a + b + c)(a + b - c)$, and cases of higher degree which can be expressed as the difference between two squares.

Case 5 is chiefly valuable in solving quadratic equations, and we shall therefore postpone the general consideration of this case until it is needed for this purpose.

It is easily seen, however, that in a simple case like $x^2 + 4x - 21$ we must find two numbers such that $a + b = 4$ and $ab = -21$. By trying the factors of -21, we easily see that $a = 7$ and $b = -3$ meet these conditions; that is, $x^2 + 4x - 21 = (x + 7)(x - 3)$.

Exercises. Factoring

1. Factor the expression $px + py - pz$ and check the work by letting $p = 1$, $x = 2$, $y = 3$, and $z = 1$. In checking, substitute the values both in the expression as given and in the factored form.

2. Factor $9\,x^2 - y^2$ and check the work by letting $x = 2$ and $y = 1$; by letting $x = 3$ and $y = 2$.

3. Evaluate $x^2 - 8\,x + 15$ for $x = 15$, first taking the expression as given and then after factoring.

Factor the following expressions, and check the work by substituting values for the letters as was done in Ex. 1:

4. $ax^2 + 2\,ax + 2\,a$.

5. $px - py + 3\,pz$.

6. $\frac{1}{2}\,hp - \frac{1}{2}\,hp' + \frac{1}{2}\,h$.

7. $mx^2 + mx - 2\,m$.

8. Factor $(x + y)^2 + 2(x + y) + 1$, $(a - b)^2 - 2(a - b) + 1$, and $x^2 + 2\,x + 1 - 2\,b(x + 1) + b^2$.

9. Factor $ax^2 + 2\,ax + a$ and $25\,n^2 - 20\,mn + 4\,m^2$.

Arrange the following expressions in forms more convenient for computation:

10. $81\,x^2 - 16$.

11. $144\,m^2 - 121\,n^2$.

12. $x^2 + 2\,xy + y^2 - z^2$.

13. $(a - b)^2 - (a + b)^2$.

14. Factor the expression $x^4 - y^4$, stating the result as the product of three binomials. Check the work by letting $x = 2$ and $y = 1$.

15. Factor the expression $6.25\,x^2 - 1.44\,y^2$.

16. Arrange $0.49\,m^2 - 0.0121$ in a form more convenient for computation, and find the value when $m = \frac{1}{49}$.

Factor the following expressions and check the work:

17. $x^2 + 2\,x - 63$.

18. $x^2 - 12\,x + 35$.

19. $m^2 + 6\,m - 55$.

20. $p^2 + 22\,p + 121$.

21. Add a term to $a^2 + 2\,ab$ that will make the expression the square of $a + b$. Make and solve a similar problem for $a^2 - 2\,ab$; for $x^2 + 2\,x$; for $4\,x^2 + 4\,x$.

22. Find the factors of $x^2 + 4\,x - 4\,y^2 + 4$, and check the work both by letting $x = 3$ and $y = 2$ and by multiplying the factors together.

23. Find the factors of $x^2y - 15\,xy^2 + 36\,y^3$, and check the work by the easiest method.

24. One factor of $x^3 + 3\,x^2y + 3\,xy^2 + y^3$ is $x + y$. Write the expression as the product of three binomial factors.

25. Since $x^4 + x^2y^2 + y^4 = x^4 + 2\,x^2y^2 + y^4 - x^2y^2$, it is evidently the difference between two squares. Write the expression as the product of two trinomial factors.

26. Proceeding in the same way as in Ex. 25, write the expression $x^4 + 9\,x^2 + 81$ as the product of two trinomial factors, and check the work by letting $x = 2$.

27. What number must be added to $x^2 + 6\,x$ so that the result is the product of two equal binomial factors? What are these two factors?

28. In the expression $ax + bx - cx = d$ find the value of x in terms of a, b, c, and d.

Factor the following expressions and check the work:

29. $4\,x^2 + 4\,x + 1$.

30. $9\,x^2 - 6\,x + 1$.

31. $49\,a^2 - 70\,a + 25$.

32. $49\,a^2 - (a + 5)^2$.

33. $(a - 7)^2 - 625$.

34. $(x + y)^2 - 49$.

35. $9\,x^2 + 30\,x + 25$.

36. $36\,m^2 - 84\,m + 49$.

37. $(a + b)^2 - (a - b)^2$.

38. $a^2 - x^2 - y^2 - 2\,xy$.

39. $4\,m^2 - 4\,n^2 + m + n$.

40. $81\,p^2 + 9\,p + 4\,q - 16\,q^2$.

41. $a^2 + bc + ab + ac$.

42. $10\,a^4 - 20\,a^3 + 10\,a^2$.

43. $4\,x^2 + 4\,y^2 - 8\,xy - 16\,z^2$.

44. $a^4 + b^4 - (4 - 2\,a^2b^2)$.

2. The Equation

Historical Considerations. Problems that we now solve by the use of equations first made their appearance, so far as we know, in Egypt about thirty-five hundred years ago. Expressed in modern form, the first of the problems was as follows: What number added to its seventh part makes 19? A few centuries later we find problems like this: Find two numbers whose sum is 20 and whose product is 51. Cases of this kind have, of course, intellectual rather than practical value. Although the solution of such problems was greatly simplified by the perfecting of our algebraic symbolism about three hundred years ago, algebra from the earliest days to the present time has always included a large number of puzzles of this type.

Practical Value of Equations. In view of the fact that so many of these problems of algebra may be said to be inherited from the earlier methods of studying the science, the question naturally arises as to whether or not the equation really has any practical value in industry, commerce, and other activities of today.

A machine designer who is planning a wheel that will travel a certain distance at each revolution needs to know the radius that will meet these conditions; that is, he needs to solve for r the equation, or formula,

$$C = 2\,\pi r.$$

The solution, of course, consists in dividing both sides by $2\,\pi$, and the result is a formula for r; namely,

$$r = \frac{C}{2\,\pi}.$$

That is, in practical cases an important application of the equation is in deriving one formula from another.

This need for the equation in the treatment of formulas has received much more attention in recent years.

General Classes of Equations. Traditionally we classify equations according to the letter for which we are solving. That is, we say that

A *linear equation* is one that contains only the first power of this letter.

For example, $ax + b = cx - d$ is a linear equation in x if, as is generally the case, we are solving for x.

Another name used for this type of equation is *simple equation*, but the term is now rarely employed.

A *quadratic equation* is one that contains the second power, but no higher power, of the letter.

For example, $4x^2 = a$ is a quadratic equation in x and $\frac{1}{2}y^2 - y + m = 0$ is a quadratic equation in y.

Although the formula $V = \pi r^2 h$, which gives the volume of a circular cylinder with the radius r and the height h, is a quadratic equation in r it is only a linear equation in h.

A *cubic equation* is one that contains the third power, but no higher power, of the letter.

If a hemispherical end is placed on a common cylinder, the formula for the volume becomes $V = \pi r^2 h + \frac{2}{3}\pi r^3$, which is a cubic equation in r and a linear equation in h.

A linear equation is also called an *equation of the first degree*; a quadratic equation is also called an *equation of the second degree*; and a cubic equation is also called an *equation of the third degree*. Similarly, we may have equations of the fourth degree, fifth degree, and so on for higher degrees.

The above types of equations are known as *algebraic equations* since they involve only the operations of addition, subtraction, multiplication, division, or the finding of powers and roots.

Later we shall take up types of equations that involve other relations than those specified above. Such equations are classified in general as *nonalgebraic equations*.

For example, $2^x = 7$ and $\sqrt[n]{32} = 64$ are types of nonalgebraic equations which have important uses.

Linear Equations. If from the formula for the area of a trapezoid,

$$A = \tfrac{1}{2}\, h(b + b'),$$

we wish to derive a formula for h, we divide the equal members, or sides, of the given formula by $\frac{1}{2}(b + b')$ and have

$$h = \frac{2\,A}{b + b'}.$$

The axiom referring to the division of equals by equals, which is usually quoted in elementary algebra, is so familiar now that such references need no longer be made.

From the same formula, if we wish to derive a formula for b, we have

$$\frac{2\,A}{h} = b + b';$$

whence

$$b = \frac{2\,A}{h} - b'.$$

The convention of writing as the left-hand member the letter for which we solve is also so familiar as to require no explanation.

Similarly, in solving for D the formula

$$H = \frac{0.02\,LV^2}{64.4\,D},$$

which is used in relation to a certain type of pump, we have

$$HD = \frac{0.02\,LV^2}{64.4};$$

whence

$$D = \frac{0.02\,LV^2}{64.4\,H}.$$

If we wish to solve the above formula for V, we have to solve a quadratic equation. The solution, which is as simple as that of a linear equation, is as follows:

From

$$H = \frac{0.02\,LV^2}{64.4\,D},$$

we have

$$V^2 = \frac{64.4\,DH}{0.02\,L};$$

whence

$$V = \pm\sqrt{\frac{64.4\,DH}{0.02\,L}}.$$

Exercises. Linear Equations

1. Solve the formula $L = 3\frac{1}{7}(R + r) + 2\,d$, which gives the approximate length of a belt around two pulleys, for d; for R.

2. From the formula $T = \dfrac{12(D - d)}{l}$, which is used in computing tapers, derive a formula for D; for d; for l.

3. From the formula $M = D_0 - pC + 3\,D_w$, which is used in measuring threads, derive a formula for p; for D_0; for C.

4. From the formula $H = \dfrac{D^2N}{2.534}$, which is used in connection with power boats, derive a formula for N; for D.

The derivation of the formula for D involves no added difficulty. In evaluating this formula, however, we have to compute a square root, but in practical work such roots are usually found from tables like those on pages 352–356. Although the double sign ($\pm$) is generally used before the square-root sign, as shown on page 11, in many cases in practical work the minus sign has no significance.

5. From the formula $T = \dfrac{33{,}000\,H}{2\,\pi r S}$, which is used in connection with revolving shafts, derive a formula for H; for r; for S.

6. From the lever formula $F = \dfrac{F'd}{2\pi l}$ derive a formula for F'; for d; for l.

7. From the formula for steel springs, $S = \dfrac{nW \times 33 \times 10^8}{r^4}$, derive a formula for n; for W; for r.

The formula for r involves a fourth-degree equation in r, and hence in evaluating it we have to find a fourth root. A little later we shall learn that a fourth root can be easily found by logarithms.

8. Solve for x the equation $\frac{2}{3}\,x + a = 3\,x - b$.

Although we conventionally say that we solve an equation and that we derive a formula from another formula, Ex. 8 is essentially the same as Exs. 1–7. Since practical work often involves formulas, problems from such sources frequently relate to derivation.

From the following formulas, which are used in physics, derive formulas, as specified:

9. $d = \frac{1}{2}gt^2$; for t.

10. $E = \frac{e(R+r)}{r}$; for r.

11. $E = \frac{1}{t}\left(\frac{T^2}{R} - 1\right)$; for R.

12. $w_1 l_1 = w_2 l_2$; for l_2.

13. $i = \frac{E}{R^2 + r^2}$; for E.

14. $C = \frac{En}{R + nr}$; for n.

15. Gay-Lussac's law for the expansion of a gas under constant pressure is often stated as $V = V_0(1 + at)$, which may be regarded as a formula for the volume V of the gas in terms of its volume V_0 at 0° C. (centigrade), the temperature t in degrees centigrade to which it is heated, and the coefficient of expansion a, which is a constant. Find a formula for t, taking the value of a as 0.00367.

16. In the formula of Ex. 15 show how the value of the constant a may be found when the values of V, V_0, and t have been determined.

17. From the formula $R = R_0(1 + at + bt^2)$, which gives the electric resistance R of a platinum wire in terms of the resistance R_0 at 0° C., the temperature t to which the wire is heated, and the constants a and b, derive a formula for finding the value of b when the values of R, R_0, a, and t have been determined.

18. Boyle's law for a perfect gas is that the product of the pressure and the volume is constant. Letting P represent the pressure, V the volume, and k a constant, express this law as a formula, and from it find the value of V when $P = 30$ and $k = 2745$.

In actual work in the physics laboratory we should, of course, know whether we were measuring P in pounds per square inch or in grams per square centimeter, and so on. The generality of the algebraic formula, however, is here illustrated by the fact that the one formula covers all such cases.

Simultaneous Equations. Suppose that in designing wire-wrapped cables it is necessary to find the length of wire per turn needed to go round a core having a certain cross-section area; that is, to find the circumference of a circle in terms of its area. Taking the basal formulas for the circle,

$$C = 2\pi r \tag{1}$$

and

$$A = \pi r^2, \tag{2}$$

we have, from (2),

$$r^2 = \frac{A}{\pi};$$

whence

$$r = \pm\sqrt{\frac{A}{\pi}}.$$

Substituting the positive value of r in (1), we then have

$$C = 2\pi\sqrt{\frac{A}{\pi}} = 2\sqrt{\frac{\pi^2 A}{\pi}},$$

or

$$C = 2\sqrt{\pi A}.$$

We thus have a formula for C which does not contain r, or, as we say, we have *eliminated* r.

Regarded as an algebraic expression, the above formula for C is said to be in its simplest form. In practical work we shall find that it is frequently easier to evaluate if we write it as $C = \sqrt{4\pi A}$.

In elementary algebra the methods of solving two *simultaneous equations*, that is, two equations which are satisfied by a common pair of values for the unknown quantities, are generally learned by taking cases like

$$2x + y = 17$$

and

$$3x - 4y = 9,$$

where one unknown is easily eliminated by addition or subtraction. In actual practice we more often find cases like the one above, or we need to solve a pair of equations like

$$1.07x + 0.62y = 1.73$$

and

$$2.06x + 5.01y = 8.34.$$

Simultaneous Linear Equations. In solving a pair of equations like

$$1.07\,x + 0.62\,y = 1.73 \qquad (1)$$

and

$$2.06\,x + 5.01\,y = 8.34, \qquad (2)$$

which were given on page 14, it is generally better to find the value of one unknown in terms of the other from one equation, and then to substitute this value in the other equation; that is, from (1) we have

$$x = \frac{1.73 - 0.62\,y}{1.07}. \qquad (1')$$

Substituting this value of x in (2), we then have

$$2.06\left(\frac{1.73 - 0.62\,y}{1.07}\right) + 5.01\,y = 8.34,$$

from which, clearing of fractions, we have

$$2.06 \times 1.73 - 2.06 \times 0.62\,y + 1.07 \times 5.01\,y = 1.07 \times 8.34.$$

Finding the result of each multiplication only to the nearest 0.01, we then have

$$4.08\,y = 5.36;$$

whence

$$y = 1.31.$$

The value of x may now be found from equation (1').

The roots found in this way are, of course, only approximate (in this case, to the nearest 0.01), but for practical purposes this is sufficient.

Equations involving decimals, like those given above, are found in the problems of physics or in certain phases of engineering work. Occasionally problems involving three linear equations in three unknowns are found in applied mathematics, but they are comparatively rare. Such cases are readily solved by first eliminating one unknown from the first and second equations and then eliminating the same unknown from the first and third (or second and third) equations. The resulting pair of equations in two unknowns is then solved by the usual methods.

Exercises. Linear Equations

1. Solve the equation $7x - 3.4 = 2x + 28.4$. Check the work by substituting the value of x in the given equation.

2. Solve the equation $4x + 0.73 = 2\frac{1}{2}x + 6.67$. Check the work as in Ex. 1.

3. Solve the equation $0.73x - 6.84 = 9.11x - 0.37$, and check the work.

In these exercises, which are typical of practical cases, the results of any operations with the decimals should be found only to the same number of places as the decimals in the given equation. The check, of course, will then be only approximate.

Solve the following pairs of equations:

4. $4x - 3y = 24$
$2.1x + 5y = -13.7$

5. $4x + 7y = 29.4$
$5x - y = 7.5$

6. $1.37x - 0.23y = 1.68$
$2.43x + 1.72y = 4.93$

7. $3x + 2y = 7.43$
$x - 7y = -2.89$

8. $\frac{2}{3}x + \frac{3}{7}y = 12$
$0.1x + \frac{1}{2}y = 7.9$

9. $2.71T + 1.07V = 3.81$
$0.62T - 3.81V = -0.68$

Exs. 4, 5, 7 and 8 are more like the equations usually studied in elementary algebra, while Exs. 6 and 9 represent the type more often found in applied mathematics.

10. From the two circle formulas $A = \frac{1}{2}Cr$ and $C = 2\pi r$ derive a formula for A in terms of r.

11. From the circle formulas $A = \pi r^2$ and $d = 2r$ derive a formula for A in terms of d.

12. From the formulas $V = \frac{4}{3}\pi r^3$ and $r = \frac{1}{2}d$, which relate to the sphere, derive a formula for V in terms of d, and then derive one for d in terms of V.

13. From the formulas $V = \pi r^2 h$ and $S = 2\pi r(h + r)$, which give the volume and the area of the total surface of a right circular cylinder, derive a formula for S in terms of V and h; in terms of V and r.

Quadratic Equations. If from the electricity formula

$$am^2 + bm + \frac{1}{c} = 0$$

we wish to derive a formula for m in terms of a, b, and c, we have to solve a quadratic equation in m. Before undertaking a case like this, let us first consider a case that will be familiar from previous work in elementary algebra; for example, let us take the equation

$$x^2 + 6x - 27 = 0.$$

There are several simple ways of solving such an equation; for example, if we factor the first member, we have

$$(x + 9)(x - 3) = 0.$$

Since the product of these two factors is zero, one of these factors at least must be equal to zero; that is, either

$$x + 9 = 0,$$

whence $x = -9$;

or $x - 3 = 0,$

whence $x = 3.$

Instead of solving by factoring, we may write the equation as

$$x^2 + 6x = 27.$$

Then, using the fact that $a^2 + 2ab + b^2 = (a + b)^2$, we see that if we add 9 to both members, the first member becomes the square of $x + 3$; that is, we first have

$$x^2 + 6x + 9 = 36,$$

and, taking the square root of both members, we then have

$$x + 3 = \pm 6;$$

whence $x = 3$ or -9.

This method is called *completing the square*; that is, we complete the square of the left-hand member by adding the square of half the coefficient of x to both members of the equation.

Exercises. Quadratic Equations

1. Find a number such that its square increased by 6 is equal to five times the number.

Such an easy puzzle relating to numbers is, of course, given only as an illustration of a simple problem involving a quadratic equation.

2. Solve the equation $(x + 2.3)(x - 0.92)(x + 43.08) = 0$.

Solve the following equations by factoring:

3. $x^2 - 7x - 18 = 0$.

4. $P^2 - 11P + 30 = 0$.

5. $T^2 + 35 = 12T$.

6. $M^2 + 5M - 14 = 0$.

7. $K^2 - 11K + 18 = 0$.

8. $2W + W^2 = 143$.

Solve the following equations by completing the square:

9. $x^2 - 2x = 35$.

10. $x^2 - 6x = 72$.

11. $x^2 - x - 3 = 0$.

12. $x^2 - 18x = -77$.

13. $T^2 - 4T = 117$.

14. $K^2 + K = 90$.

15. $\frac{1}{4}t + t^2 = 17$.

16. $M^2 + 3M = -2$.

17. If cutting a strip 2 in. wide from one side of a square plate of sheet metal leaves 597.96 sq. in. in the plate, what was the length of the side of the original plate?

Do both roots meet the conditions of the problem? In general, only one of the roots in such cases meets practical considerations, and the other should therefore be discarded.

18. The formula $d = a + v_0t - 16t^2$ gives the approximate distance d in feet to which a body will rise in t seconds if projected upward with a velocity of v_0 feet per second from an initial height of a feet. How long will it take a rocket to reach a height of 1800 ft. if it is fired vertically upward with a velocity of 120 ft./sec. from an airplane that is 1600 ft. above the ground?

The expression 120 ft./sec. means 120 ft. per second. Abbreviations like this are common in practical work.

Are both roots in this case permissible? If so, state the reasons.

Factoring the Quadratic $ax^2 + bx + c$. In solving the equation

$$10\,x^2 + 11\,x - 6 = 0$$

by factoring it is evident that the factors of the first term must be $10\,x$ and x or $5\,x$ and $2\,x$. Possible factors of the constant term -6 are 6 and -1, -6 and 1, 3 and -2, and -3 and 2.

Since the sum of the products of the first term of each factor and the last term of the other must be 11, it is easy to find by trial that the factors which meet this condition are $5\,x - 2$ and $2\,x + 3$. That is, we have

$$(5\,x - 2)(2\,x + 3) = 0.$$

Then, either $5\,x - 2 = 0$,

whence $x = \frac{2}{5}$;

or $2\,x + 3 = 0$,

whence $x = -1\frac{1}{2}$.

The fact that $10\,x^2 = (-10\,x)(-x)$ as well as $10\,x \cdot x$ will be found to give no new factors.

Another method of obtaining the same result is first to observe that

$$(ax + b)(a'x + b') = aa'x^2 + (ab' + a'b)x + bb'.$$

We can now see that we must have $ab' + a'b = 11$ and $aa'bb' = 10 \cdot (-6) = -60$. Since the sum of the numbers 15 and -4 is 11 and their product is -60, we may proceed as follows:

$$\begin{aligned} 10\,x^2 + 11\,x - 6 &= 10\,x^2 + 15\,x - 4\,x - 6 \\ &= 5\,x(2\,x + 3) - 2(2\,x + 3) \\ &= (5\,x - 2)(2\,x + 3). \end{aligned}$$

The roots of the equation are then found as before.

In solving simple quadratic equations, such as are given in elementary algebra, the first trial method, which can be easily checked by multiplying the factors, will generally suffice. We shall soon review a better method which can be applied to all cases.

Exercises. Quadratic Equations

1. Solve the equation $(3x - 7)(5x + 2) = 0$.

This case, the solution of which is evident, illustrates the fact that the solution of a quadratic equation reduces to the solution of two linear equations. In this case one of the linear equations is $3x - 7 = 0$ and the other is $5x + 2 = 0$.

2. Factor the expression $12x^2 + x - 6$, and then solve the equation $12x^2 + x - 6 = 0$.

Solve the following equations:

3. $x^2 + 5x + 6 = 0$.

4. $x^2 - 3x - 28 = 0$.

5. $2x^2 - 5x - 3 = 0$.

6. $6x^2 - 5x - 6 = 0$.

7. $3T^2 + 11T - 4 = 0$.

8. $3P^2 + 17P - 6 = 0$.

9. Solve the equation $30x^2 + 68x + 30 = 0$.

First divide both members by 2 and then factor.

Solve the following equations:

10. $8p^2 - 42p + 10 = 0$.

11. $30T^2 + 65T + 30 = 0$.

12. How many roots does a linear equation have? Give the reasons for your answer. Why does a quadratic equation have two roots, and only two?

13. Solve the equation $0.3x^2 + 1.7x = 0$.

What monomial factor is there in the first member?

Solve the following equations:

14. $4.7x^2 - 0.9x = 0$.

15. $10m^2 + 2 = 9m$.

16. $0.32x^2 + 1.73x = 0$.

17. $x^3 + 3x^2 + 2x = 0$.

18. $P^2 + P - 12 = 0$.

19. $(x - a)^2 + x = a$.

20. $(a - x)^2 = a - x$.

21. $K(15K^2 - 16K - 15) = 0$.

22. Solve the equation $(x^2 + 8x + 12)(x^2 - 5x + 6) = 0$.

This equation, which can be solved in the same way as Ex. 1, is an equation of the fourth degree and therefore has four roots.

The Quadratic Formula. In a case involving an equation like

$$2.73\,x^2 - 0.71\,x - 1.36 = 0,$$

a type that is more likely to arise in practical work than the case considered on page 19, it would not be a simple matter either to factor or to complete the square.

From the general form of the quadratic equation,

$$ax^2 + bx + c = 0,$$

we have $$ax^2 + bx = -c,$$

and, dividing by a, $$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Completing the square of this equation, we then have

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2};$$

whence $$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a},$$

and $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We have thus developed a formula which we may use in finding the roots of any quadratic equation.

For example, in a simple case like $2x^2 - 3x - 9 = 0$ we see that we have $a = 2$, $b = -3$, and $c = -9$, and hence

$$x = \frac{3 \pm \sqrt{9 + 72}}{4} = \frac{3 \pm 9}{4} = 3 \text{ or } -1\tfrac{1}{2}.$$

Applying the formula to the equation at the top of the page, we have

$$x = \frac{0.71 \pm \sqrt{0.71^2 + 4 \times 2.73 \times 1.36}}{2 \times 2.73}$$

$$= \frac{0.71 \pm \sqrt{15.3553}}{5.46}$$

$$= 0.85 \text{ or } -0.59, \text{ each to the nearest } 0.01.$$

The accuracy to which the result should be given would be determined by the conditions in a practical case. In such work it frequently happens that the negative result has no significance, and may be discarded.

Exercises. Quadratic Equations

1. By completing the square derive a formula for solving the special case of the quadratic, $x^2 + px + q = 0$. Then derive the same formula from the general formula given on page 21 by letting $a = 1$, $b = p$, and $c = q$.

2. From the formula derived in Ex. 1 find the roots of the equation $x^2 - 19x + 84 = 0$. Find the roots also by using the general formula given on page 21.

Using the general formula, solve the following equations:

3. $6M^2 + 5M - 6 = 0$.
4. $6t^2 + 13t + 6 = 0$.
5. $7p^2 - 37p - 6 = 0$.
6. $2K^2 - 13K - 45 = 0$.
7. $6x^2 - 3x - 45 = 0$.
8. $3n^2 - 30n + 63 = 0$.
9. $20x^2 + 23x - 21 = 0$.
10. $1.5V^2 + 5V + 1.2 = 0$.

It should be observed that in two of these cases the work can be simplified by first removing a numerical factor. If desired, the decimals in Ex. 10 may first be removed by multiplying both members of the equation by 10.

11. Find two consecutive odd numbers whose product is 2499; whose product is 7055.

Any number may be represented by n, an even number by $2n$, and an odd number by $2n + 1$.

12. Find two consecutive numbers whose product is 2550; whose product is 2142.

13. Solve the equation $E^2 - 4VE + 2V^2 = 0$ for E in terms of V; for V in terms of E.

14. Solve the equation $W^2 - 2(m - n)W - 4mn = 0$ for W.

15. From the equation $6t^3 + 11t^2 - 121t = 0$, find three values for t.

16. If a railroad train had traveled 10 mi./hr. faster, it would have taken 30 min. less to travel 100 mi. Find the speed of the train.

Checking Roots. Although it is possible to check the roots of a quadratic equation by substituting them in the equation, this would evidently be a tedious process in a case like the equation given on page 21,

$$2.73\,x^2 - 0.71\,x - 1.36 = 0.$$

If, however, we designate the two roots of the general quadratic $ax^2 + bx + c = 0$ as x_1 and x_2, we may write the formula on page 21 as

$$x_1 = \frac{-b + \sqrt{b^2 - 4\,ac}}{2\,a}$$

and

$$x_2 = \frac{-b - \sqrt{b^2 - 4\,ac}}{2\,a}.$$

From these two formulas, we then have

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4\,ac}}{2\,a} + \frac{-b - \sqrt{b^2 - 4\,ac}}{2\,a} = -\frac{b}{a},$$

and

$$x_1 x_2 = \frac{-b + \sqrt{b^2 - 4\,ac}}{2\,a} \cdot \frac{-b - \sqrt{b^2 - 4\,ac}}{2\,a}$$

$$= \frac{b^2 - b^2 + 4\,ac}{4\,a^2} = \frac{c}{a};$$

that is, *the sum of the roots of* $ax^2 + bx + c = 0$ *is* $-\frac{b}{a}$, *and the product of the roots is* $\frac{c}{a}$.

For example, in the equation $2\,x^2 - 3\,x - 9 = 0$, which we found on page 21 has the roots 3 and $-1\frac{1}{2}$, we have

$$3 + (-1\tfrac{1}{2}) = \frac{3}{2} \quad \text{and} \quad -\frac{b}{a} = -\frac{-3}{2} = \frac{3}{2};$$

also,

$$3 \times \left(-\frac{3}{2}\right) = \frac{-9}{2} \quad \text{and} \quad \frac{c}{a} = \frac{-9}{2}.$$

Similarly, in the equation $2.73\,x^2 - 0.71\,x - 1.36 = 0$, which has the approximate roots 0.85 and -0.59, we find, to the nearest 0.01, that

$$0.85 + (-0.59) = 0.26 \quad \text{and} \quad -\frac{b}{a} = -\frac{-0.71}{2.73} = 0.26;$$

also,

$$0.85 \times (-0.59) = -0.50 \quad \text{and} \quad \frac{c}{a} = \frac{-1.36}{2.73} = -0.50.$$

Exercises. Review of Equations

1. The formula $d = \frac{1}{2}\, gt^2$ gives the approximate distance d in feet through which a body starting from rest will fall in t seconds, g being a constant with the approximate value 32. How long will it take an object to reach the ground if dropped from an airship at a height of 6000 ft.?

The result is only approximate, since we disregard the air resistance.

2. The formula for the area of the total surface of a closed cylindric container is $S = 2\,\pi r(h + r)$. In manufacturing tin cans which shall have the same height but which shall require only half as much tin (half the surface area) as a can 6 in. high and having 418 sq. in. of surface, what radius should be planned for the cans?

3. From the formula $s = \dfrac{d^2 w}{8\,p}$, which is used in connection with the stretch of trolley wires, derive a formula for d.

4. From the electricity formula $am^2 + bm + \dfrac{1}{c} = 0$ derive a formula for m.

5. Solve the projectile formula $d = v_0 t - \frac{1}{2}\, gt^2$ for t.

6. From the formula $T = a + bt + ct^2$, which is used in connection with steel shafts, derive a formula for t.

7. From the electricity formula $S = 145.5\, w^2 + 120\, w$ derive a formula for w.

8. From the law of falling bodies, $d = \frac{1}{2}\, gt^2 + v_0 t$, derive a formula for t.

9. From the pulley formula $s = 0.1\, gt^2 + 2\, t + C$ derive a formula for t.

10. From the artillery formula $y = \frac{1}{4}\, x(R - x)$ derive a formula for x.

11. From the artillery formula $a = 5(K^2 + 3\, K)$ derive a formula for K.

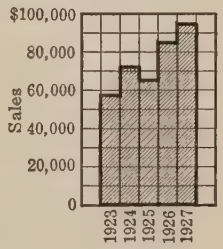
3. Graphs

Graphic Representation. One of the most interesting and important developments of modern applied mathematics has been the increased attention paid to the pictorial, or graphic, representation of statistical material of all kinds. Although such a picture may supply no more information than the statistics which it represents, it often sets forth the facts more vividly and adds materially to an appreciation of the subject under discussion.

While the graphic representation of statistics, such as crop reports, weather reports, business summaries, and the like, is not strictly a part of algebra, we shall often find it convenient to apply similar methods in representing formulas and equations to enable us to study the problems that they present. We shall therefore first discuss the graphic representation of simple statistics.

Bar Pictogram. For example, the accompanying diagram is a graphic representation of the following sales summary of a small store:

Year	Sales
1923	$58,000
1924	72,000
1925	64,000
1926	83,000
1927	95,000

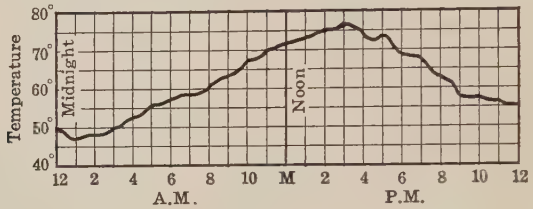


Sales Summary
1923–1927

In the diagram, which is called a *bar pictogram*, each square represents $10,000. The length of each bar thus represents the sales for that year, and the whole diagram gives a picture of the sales for the five-year period.

Pictograms like this are most conveniently drawn on *squared paper*, or *coordinate paper*, which comes already ruled in squares. It is often more convenient to have the bars placed horizontally instead of vertically, as they are in the above case.

Temperature Curve. The Weather Bureau uses thermometers which have a mechanical device for making a continuous record of the temperature. This record is traced by a pen upon a moving roll of specially-ruled paper that is unwound by clockwork at a uniform rate. The figure below shows such a record of the temperature during a 24-hour period.



Temperature Record, Oct. 5

This record, in which each vertical space represents 5° and each horizontal space represents 1 hr., shows that the temperature dropped from 50° at midnight to about 47° at 1 A.M. It then started to rise, until at 3 A.M. it again stood at 50°. This rise was continued with slight irregularities until 3 P.M., when it touched 76° and then started to fall. Between 4.30 P.M. and 5 P.M. there was a slight rise of about 2°, after which the mercury fell fairly steadily until midnight. At this time the temperature was 55°, or 5° warmer than at the preceding midnight.

Graphs. A curve like the one described above is called a *graph.* In work with graphs it is necessary to know (1) how to draw the graph from given statistics in such a way that the facts are brought out most vividly, and (2) how to read the story told by a graph that is not accompanied by the data from which it was constructed. The latter is often called *interpreting the graph.*

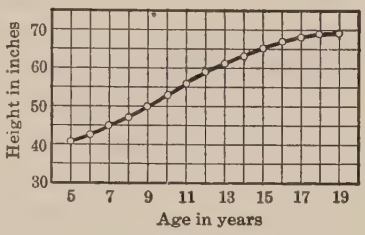
The temperature graph above was, of course, drawn mechanically, which is only possible in such special cases. The paragraph printed in small type below the diagram is the interpretation of the graph, and although a much longer story could have been told, it presents the important features.

The word "graph" is also used to include bar pictograms and other graphic methods of representing statistics.

Curve-Line Graphs. Beginning with his fifth birthday, a boy's parents kept a record of his height in inches, as follows:

Age	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Height	41	43	45	47	50	53	56	59	61	63	65	67	68	69	69

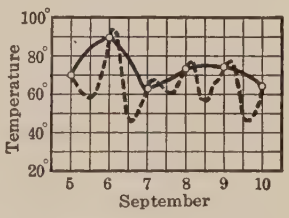
Letting the horizontal spaces represent the boy's age and the vertical spaces his height, we may locate the points corresponding to his height at each birthday as shown by the small circles in the accompanying figure. Connecting these points by a smooth curve, we have the graph here shown.



A Boy's Height, 5 yr.–19 yr.

In general, a boy's growth is distributed evenly through the year; that is, he does not increase in height only during the summer. The above graph therefore represents his increase in height during the whole period from his fifth to his nineteenth birthdays. Thus, at $9\frac{1}{2}$ yr. of age he was about $51\frac{1}{2}$ in. tall, at $12\frac{1}{2}$ yr. of age he was 5 ft. tall, and so on.

In the above case, where the growth was evenly distributed during each year, the graph gives an accurate picture of the variations in height. We may, however, be greatly deceived if this point is disregarded. For example, the solid line in this figure connects the points corresponding to the noon temperatures on six successive days, but the true story of the temperature variations is told by the dotted line, which shows how unreliable the solid line is as a picture of the situation. To represent the noon temperatures graphically, it would be better to use a bar pictogram like the one on page 25.



Temperature Record Sept. 5–10

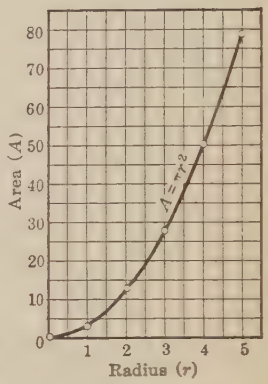
Graph of a Formula. In the graphs thus far considered each horizontal space has represented some unit of time, — a year, an hour, or a day. In statistical representation this element of time enters frequently, but in the graphs used in algebra this is not generally the case. By assigning successive values to r in the formula

$$A = \pi r^2,$$

we may compile the following statistics relating to the area of circles which have different radii:

If $r =$	0	1	2	3	4	5	· · ·
$A =$	0	$3\frac{1}{7}$	$12\frac{4}{7}$	$28\frac{2}{7}$	$50\frac{2}{7}$	$78\frac{4}{7}$	· · ·
	0	3.1	12.6	28.3	50.3	78.6	· · ·

The second row opposite A gives the decimal equivalents, which are easier to locate on the squared paper. Such a table may, of course, be continued indefinitely.



Area of a Circle

Taking the horizontal spaces for the values of r, each space representing 0.5, and the vertical spaces for the values of A, each space representing 5, we may locate the points corresponding to the six pairs of values in the table and then connect them by a smooth curve, as shown.

We thus have a graphic representation of how the area of a circle depends, not upon the time of the observation, but upon the length of the radius. Such a graph is useful not only in giving a picture of this relation, but also in enabling us to read from the graph the approximate area of a circle with a radius between 0 and 5.

For example, the area of a circle with a radius of 2.5 is seen to be approximately 20; that is, following along the vertical line corresponding to $r = 2.5$, we see that the graph cuts this line at a point corresponding to $A = 20$. If $r = 2.5$ in., we thus see that $A = 20$ sq. in.; if $r = 2.5$ cm., we see that $A = 20$ sq. cm.; and so on.

Exercises. Drawing Graphs

1. A manufacturing company found that its total sales in eight successive years were as follows: \$182,000; \$190,000; \$198,000; \$194,000; \$187,000; \$196,000; \$203,000; \$205,000. Represent these facts by a bar pictogram.

2. Represent the following statistics of a girl's height in inches on successive birthdays by a curve-line graph:

Age	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Height	40	42	45	48	50	52	53	57	60	64	65	66	67	67

3. Draw a graph representing the squares of the whole numbers from 0 to 10.

4. Draw a graph showing the cubes of the whole numbers from 0 to 5, and from it find the value of 1.5^3; of 3.5^3.

5. The length of the longest night of the year in different latitudes is shown in the accompanying table. Draw the graph and find the approximate length of the longest night in New York, which is in latitude 40° 48′ 35″ N., and find the latitudes in which the night is respectively 1 hr. longer and 1 hr. shorter than in New York.

Latitude	Length of Night	
	hr.	*min.*
5°	12	17
15°	12	53
25°	13	34
35°	14	22
50°	16	9
60°	18	30
65°	21	15

6. Draw a bar pictogram to represent the relation between the area of a regular polygon 1 in. on a side and the number of sides, as shown by the following table of values:

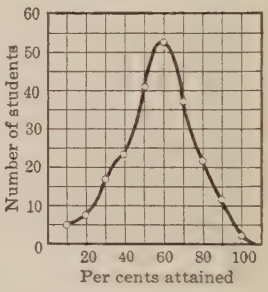
Number of sides	3	4	5	6	7	8	9	10
Area in square inches	0.4	1	1.7	2.6	3.6	4.8	6.2	7.7

7. Draw a graph of the formula $A = \frac{1}{4}\pi d^2$ for values of d between 0 and 10, and from it find the approximate area of a circle that is $3\frac{1}{2}$ ft. in diameter.

Frequency Curve. In tabulating the results of an examination which was taken by 220 students, the different grades were first arranged in ten classes or groups. The first class (or group) included all who received grades beginning with 0 and less than 15%, and was designated as the 10% class. The second, or 20%, class included all whose grades began with 15% and were less than 25%, and so on for the other groups. The result of this classification was as follows:

Class	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
Number in class	5	8	17	24	41	53	37	22	11	2

If we represent these statistics graphically, taking the horizontal spaces for the classes (or per cents), and the vertical spaces for the number of students in each class, we have the curve shown at the right.



Frequency Curve of Examination Results

Both the table and the graph show the frequency with which the various per cents were attained. The former, therefore, is called a *frequency table*, and the latter a *frequency curve*. Since the curve shows the distribution of the students according to their grades, it is also known as a *distribution curve*.

In general, it will be found that the classification of the results of a series of observations such as the case described above, or of a series relating to the height and weight of a number of persons of the same sex and of approximately the same age, is represented graphically by a bell-shaped curve like the one in the above figure. This distinctive shape is a characteristic of most frequency curves.

The increased use of frequency curves in modern investigations makes a knowledge of the fundamental principles an important asset.

Modal Class. The curve on page 30 shows that the number of students in the 60% class was greater than that in any of the other classes; that is, the 60% class in this case was the class of greatest frequency. In such a series of observations the class of greatest frequency is called the *modal class*, or simply the *mode*.

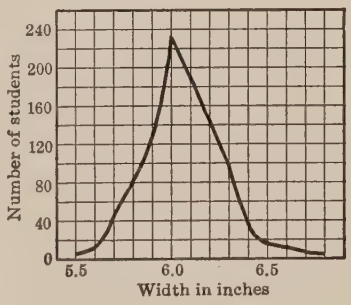
The modal class is also known as the *modal group*.

As is the case with most observations in practical work, the frequency curve on page 30 does not rise at exactly the same rate from either end towards the vertical line which corresponds to the modal class; that is, the curve is not symmetric with respect to this line.

Normal Frequency Curve. A frequency curve that is symmetric with respect to the perpendicular which represents the modal class is known as a *normal* frequency curve.

For example, the measurement of the width of the heads of 1000 university students yielded the following results:

Width in Inches	Number of Students	Width in Inches	Number of Students
5.5	3	6.2	142
5.6	12	6.3	99
5.7	43	6.4	37
5.8	80	6.5	15
5.9	131	6.6	12
6.0	236	6.7	3
6.1	185	6.8	2



Frequency Curve of Head Widths

As shown by the graph at the right, the curve which represents these results is not symmetric with respect to the vertical line representing the modal class of 6.0 in., and is therefore not a normal frequency curve. It is, however, a rough approximation to such a curve, and is about as close as is found in most cases.

Terms Used. In studying a frequency curve in order to interpret the facts which it reveals about the distribution of the results of the observations, other terms besides those given on pages 30 and 31 are often used. For example, when we speak of the *variates*, we mean the objects which are classified by the measurement that is applied to each one.

Thus, in the case given on page 30 the variates are the abilities of the students as measured by their grades in the examination. In the case given on page 31 the variates are the students' heads measured with respect to width.

When we speak of the *range*, we mean the interval between the smallest and the largest measures.

For example, in the case on page 30 the smallest class was 10 % and the largest 100 %, a range of 90 %; in the case on page 31 the range was from 5.5 in. to 6.8 in., a range of 1.3 in.

The term *arithmetic mean* is used to indicate the average found by dividing the sum of the measures of all the variates by the number of variates.

For example, in the case on page 30 the mean is 56.6 %. The statistical use of the mean is to measure the variates as a whole; that is, in this case the average grade received was 56.6 %.

In every case there is a position, usually within some class, such that half the variates lie below it and half of them lie above it. This position is called the *median*.

In the case given on page 30 half of the variates is 110, and since 95 of the variates lie below the 60 % class and 72 lie above it, the median lies in the 60 % class.

When the distribution is such that the frequency curve is approximately normal, the arithmetic mean, the median, and the modal class lie very near together.

Similarly, the *first quartile* is the position such that one fourth of the variates lie below it and three fourths above it, and the *third quartile* is the position such that three fourths of the variates lie below it and one fourth above it.

Exercises. Frequency Curves

1. A recent study of traffic accidents in New York City showed that the accidents were distributed as follows:

Time A.M.	Number of Accidents	Time A.M.	Number of Accidents	Time P.M.	Number of Accidents	Time P.M.	Number of Accidents
12–1	375	6–7	175	12–1	1175	6–7	1900
1–2	300	7–8	550	1–2	1250	7–8	1350
2–3	175	8–9	800	2–3	1350	8–9	1250
3–4	100	9–10	850	3–4	1575	9–10	850
4–5	125	10–11	1050	4–5	1700	10–11	550
5–6	150	11–12	1250	5–6	2100	11–12	450

Draw the distribution curve and write a brief statement showing the reasons for the peaks and depressions in the curve at certain hours during the period studied.

2. In Ex. 1 what is the mean number of accidents? At what time did the modal number of accidents occur?

3. In eleven successive years the per cents of candidates who received passing grades in a series of plane-geometry examinations for entrance to college were as follows: 40%, 34%, 56%, 59%, 42%, 57%, 37%, 61%, 56%, 36%, and 64%. Represent the distribution of the passing grades in these years (1) by a bar pictogram, (2) by connecting the successive points by a broken line, and (3) by a distribution curve. State which one of the three methods is the best in this case and give the reason.

4. The average of a series of laboratory experiments as to how far a ball weighing 1 oz. would fall in 1 sec. gave 16 ft. for the distance. For 1.5 sec. the average was 36 ft., for 1.75 sec. the average was 49 ft., for 2 sec. it was 64 ft., and for 2.5 sec. it was 100 ft. Represent these results by a curve and from the curve find how long it would take the ball to fall 20 ft.; 50 ft.; 80 ft.

Function. In our work with formulas in the preceding pages we found that when we knew the value of one of two quantities involved in a formula we could find the value of the other. When one quantity depends upon another in such a way that the first is determined when the second is specified, the first quantity is said to be a *function* of the second.

For example, for any specified value of r in the formula $A = \pi r^2$ a value of A is determined; that is, the area of a circle is a function of the radius. Similarly, the formula $A = s^2$ shows that the area of a square is a function of the length of a side. Furthermore, since for a specified value of A we can determine a value of s, the side is a function of the area.

Functional Relations. There are two types of functional relations, one of which admits of precise mathematical formulation and the other not. For example, the mathematical formulation of the functional relation of the area of a circle to the radius is stated by the formula $A = \pi r^2$. On the other hand, although the graph on page 26 shows that the temperature is a function of the time (that is, the temperature depends upon the hour at which it is observed) it is impossible to formulate this relation mathematically.

Function Symbols. If we were making a series of experiments with falling objects in which the distance fallen is a function of the time, it would be inconvenient to write in each case "the distance fallen in — seconds is — feet." Instead we may agree that d represents the distance fallen in feet and t the time in seconds, and we may then use the symbol $d(t)$, which is usually read "d of t" and means that d is a function of t. Then if we write $d(1) = 16$, we mean that the distance fallen in 1 sec. is 16 ft. Similarly, we may write $d(2) = 64$, and so on for other values of d as determined from the law of falling bodies, $d = \frac{1}{2} gt^2$.

In the same way we use the symbols $f(\ \ )$, $F(\ \ )$, and so on, to indicate functions: for example, $f(x) = x + 1$ means that $x + 1$ is a function of x.

Exercises. Functions

1. Since $x^2 - 3$ depends for its value upon the value of x, we may write $f(x) = x^2 - 3$. Find the value of $f(2)$; that is, find the value of $f(x)$ when $x = 2$. Find also the value of $f(-1)$; of $f(0.1)$; of $f(0)$.

2. A series of observations of a railroad train showed that the distance of the train (in miles) from its starting point at successive intervals (in minutes) was as follows:

Time	0	2	4	6	8	10	12	14	16	18	20
Distance	0	0.5	1.5	2.8	4.4	6.5	8.2	10	12.2	13.4	16

Representing the time by horizontal spaces and the distance by vertical spaces, draw a graph to illustrate the functional relation between the distance d and the time t. From the graph find the approximate value of $d(4.5)$; of $d(14.5)$; of $d(19.25)$.

3. Draw a graph of the functional relation represented by $f(x) = x^2 + 1$ for positive values of x, and find from the graph the value of $f(5)$. Check your result by finding the value numerically.

In drawing the graph locate points for integral values of x from 0 to 6 inclusive, taking the horizontal spaces for the values of x. The question of negative values of x in such cases is discussed later.

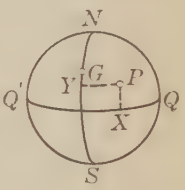
4. Draw a graph of the functional relation represented by $C(g) = 0.18\,g$, which gives the cost C of g gallons of gasoline at 18¢ per gallon. From the graph find the value of $C(5)$; of $C(15)$; of $C(8\frac{1}{2})$.

Such graphs are helpful in saving computation.

5. Write the meaning of the symbol $P(m)$, where P represents the number of pounds of coal used per minute by a train traveling m miles per hour, and also give the meaning of the expression $P(16) = 11$.

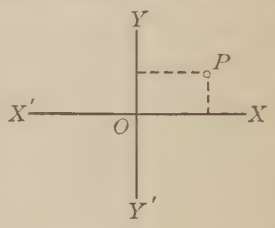
Axes. In the graphs that we have been considering we have seen how easily a point may be located on squared paper by using the horizontal and vertical spaces. Although the same principles are used in drawing the graphs of algebraic equations and formulas, it is convenient in studying the graphs to employ certain technical terms. Since these terms may not be familiar to all students, they will now be explained.

We know from geography that we locate a point on the surface of the earth by stating its distance in degrees east or west of an arbitrarily-chosen prime meridian and its distance in degrees north or south of the equator. The first distance is called the longitude of the point and the second is called the latitude.



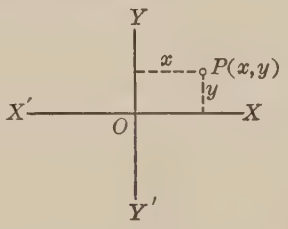
For example, taking NGS in this figure as the prime meridian (0°) through Greenwich, England (G in the figure), and $Q'Q$ as the equator, we may locate the position of a point P with reference to these two lines. Thus, if $YP = 40°$ and $XP = 30°$, we say that the position of P is 40° E. and 30° N.

Similarly, we may locate a point in a plane by taking two lines at right angles to each other and stating the distance of the point from these two lines. The two lines which are at right angles to each other are called the *axes of reference*, or simply the *axes*. The two axes are usually lettered as here shown, and their point of intersection (O) is called the *origin*. The vertical axis is called the y axis, and the horizontal one is called the x axis. The distances from the axes to a point like P are then measured along lines parallel to the axes, as shown by the dotted lines in the figure.



While the axes might have been taken oblique to each other, for our purposes we shall need to consider only cases in which the axes are at right angles to each other.

Coordinates. The distance of a point P from the y axis is called the *abscissa* of the point. Since this distance can be measured along the x axis, it is usually represented by the letter x. The distance of the point from the x axis is called the *ordinate* of the point, and is usually represented by the letter y. Taken together, the abscissa and the ordinate are called the *coordinates* of the point.

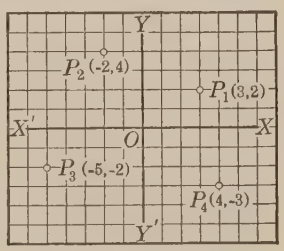


The position of a point in the plane may thus be briefly and accurately described by specifying its coordinates. Conversely, if the coordinates of a point are specified, its position in the plane may be located, or plotted, by laying off the given distances from the axes.

For these reasons we generally speak of a point by referring to its coordinates. Thus, when we speak of the point $P(x, y)$ we mean the point P which has the abscissa x and the ordinate y.

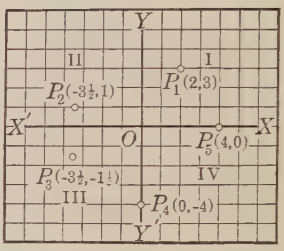
For example, we see that $P(2, 3)$ is the point P for which $x = 2$ and $y = 3$; that is, it is 2 units to the right of the y axis and 3 units above the x axis. Similarly, $P(6, 2)$ is a point which is 6 units to the right of the y axis and 2 units above the x axis.

Abscissas to the right of the y axis are considered positive and those to the left are considered negative. Similarly, the ordinates above the x axis are considered positive and those below are considered negative.



For example, the four points P_1, P_2, P_3, and P_4 here shown have the following coordinates: P_1, 3 and 2; P_2, -2 and 4; P_3, -5 and -2; P_4, 4 and -3, the coordinates in each case being given in the usual order x, y. Instead of using P_1, P_2, P_3, and P_4, we may simply designate the four points as (3, 2), $(-2, 4)$, $(-5, -2)$, and $(4, -3)$.

Quadrants. We also speak of the four sections into which the plane is divided by the axes as the four *quadrants*. We number these quadrants I to IV, taking the quadrant formed by the lines OX and OY as I and proceeding in counterclockwise order as shown in the accompanying figure.



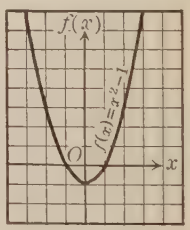
We thus see that in the four quadrants we have the following arrangement of signs:

Quadrant	Abscissa	Ordinate
I	+	+
II	−	+
III	−	−
IV	+	−

For example, the point $P_1(2, 3)$ in the above figure is in quadrant I, the point $P_2(-3\frac{1}{2}, 1)$ is in quadrant II, and the point $P_3(-3\frac{1}{2}, -1\frac{1}{2})$ is in quadrant III. That is, if the point $P(x, y)$ is in quadrant I, both x and y are positive; if in quadrant II, x is negative and y positive; if in quadrant III, both x and y are negative; and if in quadrant IV, x is positive and y negative.

When either x or y is zero, we have a point on one of the axes as shown by the point $P_4(0, -4)$ on the y axis and the point $P_5(4, 0)$ on the x axis. The origin O is always the point $(0, 0)$.

Function Axis. In locating the points corresponding to the values of a function for different values of the varying quantity, we represent the values of the quantity on the x axis and the values of the function on the y axis.



For example, in $f(x) = x^2 - 1$, where the values of $f(x)$ depend upon the values of x, we represent the values of x on the x axis and the values of $f(x)$ on the y axis as here shown.

In such a case the y axis is called the *function axis*, and it is often more convenient to designate the axes as shown.

Exercises. Plotting Points

1. Using coordinate paper as the basis for a Mercator's projection of the earth's surface, draw one axis to represent the equator and another to represent the prime meridian, and then locate the points 30° E., 20° N.; 40° W., 30°N.; 50° W., 20° S.; 50° E., 30° S.

In a Mercator's projection the earth is considered flat and the meridians are straight lines perpendicular to the equator.

2. Plot the points (30, 20), (− 40, 30), (− 50, − 20), and (50, − 30), and state in which quadrant each point lies.

The results should be compared with those of Ex. 1.

3. How far from the origin is the point (3, 4)? the point (− 3, 4)? the point (− 4, − 3)?

4. Plot the point having the abscissa 7 and the ordinate − 4; the abscissa − 4 and the ordinate 7.

5. Plot four points having positive coordinates with the abscissas respectively equal to their ordinates.

6. Solve Ex. 5 for negative coordinates.

7. In $f(x) = x + 4$, what is the value of $f(3)$? Plot the point (x, y) for which $x = -3$ and $y = f(x)$.

8. In what quadrants are the points (− 2, − 6) and (4, − 2)? Name another point in each of these quadrants.

9. Where is the point (0, 0)? the point (5, 0)? the point (0, 5)? the point (− 5, 0)?

10. What is the distance from (4, 0) to (− 4, 0)?

11. What is the distance from (4, 3) to (− 4, 3)? from (4, 3) to (4, − 3)? from (0, 0) to (0, 7)?

12. Plot the points (− 2, 3), (− 4, 6), and (− 5, 2).

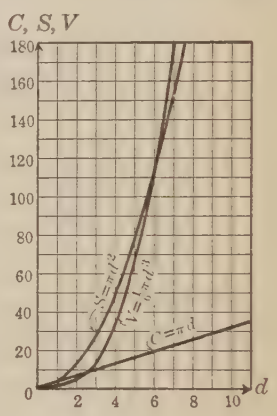
13. Plot the points (2, − 3), (3, − 2), (4, − 6), (6, − 4), (7, − 7), (− 7, 9), (0, − 4), $(1\frac{1}{2}, 2\frac{1}{4})$, and (− 0.5, − 3).

Graphs of Formulas. On page 28 we learned the advantages of representing a formula by a graph. It is often convenient to draw on the same axes the graphs of two or more related formulas; for example, the formulas

$$C = \pi d, \quad S = \pi d^2, \quad \text{and} \quad V = \tfrac{1}{6}\pi d^3$$

give respectively the circumference C of a section through the center of a sphere, the area S of the surface of the sphere, and the volume V, d in each case being the diameter.

Taking the values of d on the x axis and the values of C, S, and V on the y axis, we may draw the three graphs here shown.



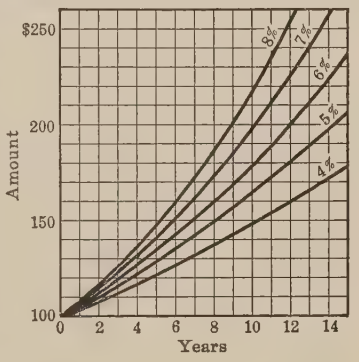
It is easy to see from the graphs that although all three functions of d increase as d increases, the circumference is the only one that increases at a uniform rate. It is also evident that, except for very small values of d, the area and volume increase more rapidly than the circumference, and that although the area increases more rapidly than the volume for values of d between 0 and 6, the volume increases more rapidly than the area after d reaches the value 6.

In order to provide maximum capacity with minimum surface area, a gas tank in Arizona was recently built in the form of a sphere. The above graphs answer many questions related to such a tank. For example, we see that while a diameter of 30 ft. (that is, of 10×3 ft.) would give a capacity of about $10^3 \times 14$ cu. ft., or 14,000 cu. ft., a diameter of 60 ft. would give a capacity of about 113,000 cu. ft. Since the area increases less rapidly after $d = 6$, if a diameter larger than 60 ft. could be used, the capacity could be greatly increased without increasing the surface area in the same proportion. At the same time, the graph of $C = \pi d$, shows that the amount of riveting to be done on a circumferential seam increases regularly as the diameter is increased.

Exercises. Graphs of Formulas

1. On the same axes draw the graphs of the formulas $E = 12\,e$, $T = 6\,e^2$, and $V = e^3$, which give the sum E of the lengths of the edges of a cube, the area T of the total surface, and the volume V. Write a statement comparing the rates at which E, T, and V increase as e increases.

2. The accompanying figure shows how an investment of \$100 increases itself at different rates when the interest as it comes due is reinvested at the same rate. From the graphs find the approximate number of years in which the investment will double itself at 5%; at 6%; at 7%; at 8%.



The figure shows the graphs of the formula $A = P(1 + r)^n$ when $P = 100$ and r has in turn the values 0.04, 0.05, $\cdots$, 0.08. That is, the graph marked 4% is the graph of $A = 100 \times 1.04^n$ for values of n from 0 to 15.

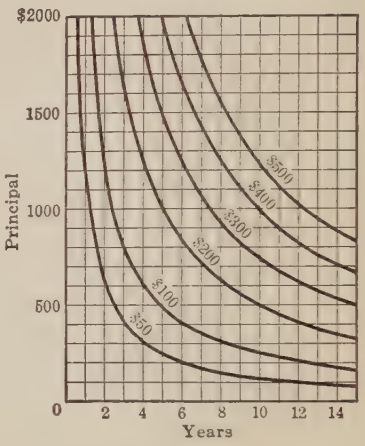
3. From the graphs of Ex. 2 determine the approximate amount of interest paid in 10 yr. on an investment of \$100 at each of the five rates of interest.

4. On the same axes draw the graphs of the formula $V = \frac{1}{3}\,\pi r^2 h$, which gives the volume of a circular cone, first when $h = 3$ while r varies from 1 to 5, and then when $r = 3$ and h varies from 1 to 5. Write a brief statement of what the graphs reveal about the rates of increase of V.

5. On the same axes draw the graphs of $y = x^2$ and $y = x^3$ for values of x from -5 to $+5$. Show how such graphs may be used to find the square roots and cube roots of small numbers like 2.5 or 3.8. How does the graph show that a negative number has no real square root?

Interest Graph. Since the interest graph on page 41 is based on a principal of \$100 and requires that the interest when due be at once reinvested, it would not help us in solving a problem that requires finding how long it would take \$400 to earn \$100 if invested at simple interest at 4%. An approximate answer to such a problem, which is typical of many that arise in business, can be found from the graph below.

When $r = 0.04$, the simple-interest formula $i = prt$ becomes $i = 0.04\,pt$. By assigning the successive values \$50, \$100, $\cdots$, \$500 to i, we have formulas involving p and t for which we can locate points that determine the curves here shown.



Following along the horizontal line for a principal of \$400 in the graph, we see that it cuts the curve marked \$100 at a point corresponding to $6\frac{1}{4}$ yr.; that is, it will take $6\frac{1}{4}$ yr. for \$400 to earn \$100 in interest at 4%.

Similarly, to find from the graph the principal which must be invested at 4% in order to yield \$300 in interest in 10 yr., we follow along the vertical line for 10 yr. and find that it cuts the curve marked \$300 at a point corresponding to \$750; that is, it will require a principal of \$750 to yield the specified interest.

The results from so small a graph are naturally only approximate in most cases. As a convenient check on a computation, or in cases where an approximate result only is desired, such graphs are a very convenient means of saving time and labor. The purpose here, of course, is simply to illustrate the general principles of graphs of this type. The device is much more useful as an aid to computation if the graphs are drawn to a sufficiently large scale and enough points are located to determine the curves accurately.

Exercises. Review

1. From the graph on page 42 find the approximate interest at 4% on \$1400 for 9 yr.; on \$1250 for 6 yr.

The horizontal line for \$1400 and the vertical line for 9 yr. intersect very close to the curve marked \$500. Hence the interest is about \$500, which is actually \$4 smaller than the correct result. A student who gave the interest as \$50.40, due to careless placing of the decimal point, would find the approximate result from the graph sufficient to reveal his error.

Using the 4% interest graph on page 42, find approximately

2. The interest on \$900 for 9 yr.; on \$1300 for 7 yr.; on \$1500 for 8 yr.

In the first case the lines for \$900 and for 9 yr. intersect at a point about one fourth of the distance between the curves marked \$300 and \$400. Hence the interest is about \$325, although accurately it is \$324.

3. The principal necessary to yield \$200 in 5 yr.; \$400 in 12 yr.; \$450 in 8 yr.

4. The time it would take \$1200 to earn \$200; \$1500 to earn \$500; \$1250 to earn \$300.

5. Construct a 5% interest graph similar to the 4% graph on page 42; that is, construct a graph of the formula $i = 0.05\,pt$ for the values \$50, \$100, \$200, \$300, \$400, and \$500 for i.

Using the graph constructed in Ex. 5, find approximately

6. The principal necessary to yield \$200 in 5 yr.; \$300 in 10 yr.; \$450 in 8 yr.

7. The time it would take \$1100 to earn \$300; \$1400 to earn \$400; \$1050 to earn \$500.

8. The interest on \$1400 for 5 yr.; on \$1800 for 3 yr.; on \$300 for 15 yr.

9. The time in which \$1000 will earn \$250; will earn \$325.

The course of the curves for \$250 and \$325 should be estimated.

10. Draw a graph to show the rainfall in a city in which the recorded fall each month was as follows: Jan., 5 in.; Feb., 1.5 in.; Mar., 4.9 in.; Apr., 2.3 in.; May, 4.8 in.; June, 3.5 in.; July, 0.7 in.; Aug., 3.2 in.; Sept., 1.5 in.; Oct., 1.8 in.; Nov., 1.6 in.; Dec., 3.6 in.

11. Draw a bar graph to show the temperature variations in a northern city in which the noon temperatures on ten successive days were as follows: 40°, 30°, 5°, − 10°, − 5°, − 20°, 0°, 10°, 50°, 45°. Why is a bar graph the preferred method of representing such statistics?

12. The distance d traveled in t seconds by an automobile having an initial velocity of v_0 feet per second and a uniform acceleration of a feet per second is given by the formula $d = v_0t + \frac{1}{2}at^2$. Draw a graph of this formula when $v_0 = 20$ and $a = 5$, taking the values $5, 10, \ldots, 25$ for t. From the graph estimate the distance traveled by the car in $10\frac{1}{2}$ sec.

13. A formula for the weight W (in pounds) of a hollow cast-iron cylindric column l feet long with an outside diameter of D inches and an inside diameter of d inches is $W = 9.80\,l(D^2 - d^2)$. Draw a graph to show the variations in weight of 10-foot columns with an outside diameter of 6 in. for different thicknesses of the metal, taking the inside diameters as 2 in., 2.5 in., 3 in., 3.5 in., 4.0 in., 4.5 in., and 5 in. From the graph estimate the weight of such a column with an inside diameter of 4.8 in.

14. The formula $F = 1.8\,C + 32$ is used in converting readings on the centigrade thermometer (C) to the corresponding readings on the Fahrenheit scale (F). Draw a graph of this formula, letting each space on the x and y axes represent 2°, and from the graph find the Fahrenheit readings that correspond to 32° C.; 0° C.; 25° C.; − 10° C.; − 25° C.

15. From the graph of Ex. 14 find the centigrade readings that correspond to 68° F.; 25° F.; 40° F.; − 5° F.

4. Generalized Exponents

Extension of the Ideas of Exponents. The early ideas about exponents required that an exponent should be a positive integer. That is, since an expression like 2^3 was considered to mean that 2 is taken three times as a factor, it was regarded as impossible to give a definite meaning to 2^0, $2^{\frac{1}{3}}$, $2^{\frac{2}{3}}$, or 2^{-3}, because we obviously cannot take 2 zero times, one third of a time, two thirds of a time, or minus three times as a factor.

With the growth of algebra, however, it was found not only possible but also desirable to give a meaning to such forms of exponents. Furthermore, it was found that these exponents and an extension of the ideas about the meanings of the terms "power" and "root" were very helpful in the solution of the problems of modern science and in the development of mathematics itself.

We therefore speak today of "a to the two thirds power," of "a to the minus nth power," of "a to the zero power," and so on, although such forms were meaningless under the earlier restrictions upon exponents.

It is only within about three centuries that a number like -2, which was meaningless to the ancients, has been fully understood. Even now, an expression like $\sqrt{-2}$, which is called an *imaginary number*, has no meaning to many people, although the student who goes on in mathematics and science will find that these imaginary numbers have a definite meaning and use.

We shall now consider the significance of the types of exponents mentioned above, and we shall see that they can be defined in such a way that they shall be in harmony with the earlier laws that were already developed with reference to positive integral exponents.

Such harmony is, of course, essential in order to avoid having one set of laws that can be applied only to cases like a^2 and m^3, and another that must be applied to $a^{\frac{2}{3}}$ and $m^{-\frac{3}{4}}$.

Fractional Exponents. The laws of positive integral exponents are already familiar; for example, we know that

$$a^2 \cdot a^3 = (a \cdot a)(a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a = a^5,$$

or

$$a^2 \cdot a^3 = a^{2+3} = a^5.$$

A general expression of this law is therefore

$$a^m \cdot a^n = a^{m+n}.$$

Similarly, $(a^3)^2 = a^3 \cdot a^3 = a^{3+3} = a^6 = a^{3\times 2}$,

or, in general, $(a^m)^n = a^{mn}$.

Thus, if a fractional exponent is to be in harmony with the law

$$a^m \cdot a^n = a^{m+n},$$

we must have $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$

Since $\sqrt{a} \cdot \sqrt{a} = a,$

we therefore agree that the expression $a^{\frac{1}{2}}$ shall have precisely the same meaning as $\sqrt{a}$.

Similarly, since $(a^m)^n = a^{mn},$

we must have $(a^{\frac{1}{3}})^2 = a^{\frac{2}{3}},$

and we therefore agree that $a^{\frac{2}{3}}$ shall have the same meaning as $(\sqrt[3]{a})^2$ or as $\sqrt[3]{a^2}$. We thus see that

In a fractional exponent the numerator indicates a power and the denominator indicates a root.

That is, stated in symbolic form,

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}.$$

For example, $9^{\frac{1}{2}} = \sqrt{9} = 3,$

$$8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = \sqrt[3]{8^2} = \sqrt[3]{64} = 4,$$

and $x^{\frac{4}{3}} = (\sqrt[3]{x})^4 = \sqrt[3]{x^4} = x\sqrt[3]{x}.$

We may therefore write expressions involving roots either with the radical sign or with fractional exponents. In simple cases like $\sqrt{2}$ and $\sqrt[3]{8}$ it is easier to use the radical sign, but in a complicated case it is frequently better to use fractional exponents.

Exercises. Fractional Exponents

1. A laboratory manual gives the formula $t = 2\pi\left(\frac{l}{g}\right)^{\frac{1}{2}}$ for finding the time of vibration t of a pendulum in seconds. Solve the formula for l and evaluate the result for $g = 32.2$ and $t = 2\frac{1}{2}$.

This formula gives the length in feet (l) of a pendulum that vibrates in $2\frac{1}{2}$ sec. The value of g varies with the latitude and with the altitude above sea level.

2. Solve the formula given in Ex. 1 for g and evaluate the result for a pendulum that is 3 ft. long and makes 400 vibrations in 12 min. 50 sec.

3. A formula for the area S of the surface of a sphere with the volume V is $S = (6V)^{\frac{2}{3}}\pi^{\frac{1}{3}}$. Express this formula with the common radical signs and show that $S = 4.84\sqrt[3]{V^2}$, approximately.

Write the following expressions with radical signs:

4. $m^{\frac{3}{8}}$. **5.** $2^{\frac{1}{2}}x^{\frac{1}{2}}$. **6.** $7^{\frac{2}{3}}x^{\frac{2}{3}}$. **7.** $x^{\frac{1}{a}}y^{\frac{1}{b}}z^{\frac{1}{c}}$.

Simplify each of the following expressions:

8. $125^{\frac{1}{3}}$. **11.** $(-32a)^{\frac{2}{5}}$. **14.** $125^{\frac{2}{3}} \cdot 625^{\frac{1}{2}}$.

9. $125^{\frac{2}{3}}$. **12.** $(-32a)^{\frac{3}{5}}$. **15.** $121^{\frac{1}{2}} \cdot 144^{\frac{1}{2}}$.

10. $125^{\frac{4}{3}}$. **13.** $(-32a)^{\frac{4}{5}}$. **16.** $\left(\frac{1}{64}\right)^{\frac{1}{2}}\left(\frac{1}{1331}\right)^{\frac{2}{3}}$.

Write the following expressions with fractional exponents in their simplest forms:

17. $\sqrt[4]{a^3b}$. **19.** $5^2 \cdot \sqrt[3]{m^2} \cdot \sqrt[4]{n^5}$. **21.** $\sqrt[n]{a^m} \cdot \sqrt[p]{b^q}$.

18. $\sqrt[4]{3a^3b^5}$. **20.** $2a^2\sqrt[4]{p}\sqrt[5]{q^2}$. **22.** $\sqrt[n]{a^{2m}} \cdot \sqrt[p]{b^qc^r}$.

23. The formula for the volume V of a cube in terms of T, the area of the total surface, is $V = (\frac{1}{6}T)^{\frac{3}{2}}$. Express this formula with the radical sign and evaluate the result for $T = 7.2$ sq. in.

Negative Exponents. In a case of division involving exponents we already know that

$$\frac{a^5}{a^2} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} = a^3,$$

or

$$\frac{a^5}{a^2} = a^{5-2} = a^3.$$

A general expression of this law is therefore

$$\frac{a^m}{a^n} = a^{m-n}.$$

Since we have defined fractional exponents, it is no longer necessary to restrict the above law to integral values of m and n. When n is numerically greater than m, however, the exponent of the quotient is negative. We thus see that it is necessary to give a meaning to a negative exponent, and, if the above law is to be general, we must have

$$\frac{a^2}{a^5} = a^{2-5} = a^{-3}.$$

Since

$$\frac{a^2}{a^5} = \frac{a \cdot a}{a \cdot a \cdot a \cdot a \cdot a} = \frac{1}{a^3},$$

we therefore agree that a^{-3} shall have the same meaning as $\frac{1}{a^3}$; that is, we agree that

A negative exponent indicates the reciprocal of the expression with the same positive exponent;

or

$$a^{-n} = \frac{1}{a^n}.$$

For example, $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$, $a^{-4} = \frac{1}{a^4}$, $a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}}$, and $a^{\frac{1}{2}}b^{-\frac{2}{3}} = \frac{a^{\frac{1}{2}}}{b^{\frac{2}{3}}}$.

In the theory of exponents, with which we are not concerned at present, it is shown that we may operate with such exponents according to the ordinary laws of fractions. It is often easier to use fractional and negative exponents instead of the common radical signs, as in the following case:

$$\sqrt[4]{\frac{\sqrt{x^3}}{\sqrt[3]{x^4}}} = (x^{\frac{3}{2}} \cdot x^{-\frac{4}{3}})^{\frac{1}{4}} = (x^{\frac{3}{2}-\frac{4}{3}})^{\frac{1}{4}} = (x^{\frac{1}{6}})^{\frac{1}{4}} = x^{\frac{1}{24}}, \text{ or } \sqrt[24]{x}.$$

Exercises. Negative Exponents

1. The formula $Q = 1000\ H^{\frac{1}{2}}d^{\frac{5}{2}}g^{-\frac{1}{2}}l^{-\frac{1}{2}}$ is used in finding the rate at which coal gas flows through a pipe. Express the formula without using fractional or negative exponents.

In this case it will be seen that the resulting form is the simpler.

2. Express the pendulum formula $t = 2\pi\sqrt{\frac{l}{g}}$ with a negative exponent in order to avoid the fractional form under the radical sign.

3. The radius r of a circular cylinder of height h and volume V is given by the formula $r = \sqrt{\frac{V}{\pi h}}$. Taking the value of $\pi^{-\frac{1}{2}}$ as 0.564, express the formula with fractional and negative exponents.

Express each of the following with positive exponents:

4. n^{-5}. **7.** $(\frac{5}{8})^{-1}$. **10.** $8^{-\frac{1}{3}}$. **13.** $n^{-\frac{2}{3}}$.

5. a^{-8}. **8.** $(\frac{1}{4})^{-2}$. **11.** $8^{-\frac{2}{3}}$. **14.** $(x+y)^{-1}$.

6. $(\frac{1}{3})^{-1}$. **9.** $(\frac{1}{5})^{-3}$. **12.** $27^{-\frac{2}{3}}$. **15.** $(m+n)^{-2}$.

Using negative and fractional exponents, express each of the following in integral form:

16. $\frac{1}{3}$. **21.** $\frac{3}{4}$. **26.** $\frac{1}{16}$. **31.** $\frac{1}{50}$.

17. $\frac{1}{x}$. **22.** $\frac{2}{n^2}$. **27.** $\frac{p}{\sqrt{q}}$. **32.** $\frac{x}{b\sqrt{y}}$.

18. $\frac{mn}{\sqrt{xy}}$. **23.** $\frac{a+b}{a-b}$. **28.** $\frac{p+q}{\sqrt{p-q}}$. **33.** $\frac{1}{\sqrt[3]{(x-y)^2}}$.

19. $\frac{xy}{\sqrt[3]{a^2b^2}}$. **24.** $\frac{\sqrt{ab}}{xy}$. **29.** $\frac{a-b}{\sqrt[3]{(a+b)^2}}$. **34.** $\frac{p^2-q^2}{p^2+q^2}$.

20. $\frac{p^2q^2}{\sqrt[4]{x^3y^3}}$. **25.** $\frac{a+1}{\sqrt[3]{x^2y^2}}$. **30.** $\frac{m(x+y)}{\sqrt[5]{(p+q)^3}}$. **35.** $\frac{p^2+q^2}{p^4-q^4}$.

Zero Exponent. It now remains to find the meaning that should be assigned to a^0 in order to have it conform to the laws of positive integral exponents. For example, if it is to conform to the law

$$a^m \cdot a^n = a^{m+n},$$

we must have $a^0 \cdot a^n = a^{0+n} = a^n$.

To have this true, a^0 must have the value 1; that is, in defining a^0 we agree that

$$a^0 = 1.$$

For example, $3^0 = 1$, $(\frac{1}{2})^0 = 1$, $(-2)^0 = 1$, and $2(x^0) = 2$.

Index Notation. By using exponents we can express very large or very small numbers by a convenient device known as *index notation* or *exponential notation*. For example, since

$$10{,}000{,}000 = 10^7,$$

instead of saying that the distance from the sun to the earth is 93,000,000 mi., we may say that it is $10^7 \times 9.3$ mi. Similarly, we may give the distance to the planet Neptune as $10^9 \times 2.8$ mi. instead of 2,800,000,000 mi. In the same way, since

$$0.0000001 = 10^{-7},$$

instead of saying that the wave length of sodium light is 0.000000589 m. we may give it as 5.89×10^{-7} m.

While the advantages of this notation are not so apparent here, we shall later see that it is a very helpful device in working with logarithms. In astronomy and physics the index notation is used very extensively, and it is therefore desirable to understand its significance.

The following table shows a few of the common powers of 10:

$0.0000001 = 10^{-7}$	$0.01 = 10^{-2}$	$1000 = 10^3$
$0.000001 = 10^{-6}$	$0.1 = 10^{-1}$	$10{,}000 = 10^4$
$0.00001 = 10^{-5}$	$1 = 10^0$	$100{,}000 = 10^5$
$0.0001 = 10^{-4}$	$10 = 10^1$	$1{,}000{,}000 = 10^6$
$0.001 = 10^{-3}$	$100 = 10^2$	$10{,}000{,}000 = 10^7$

Exercises. Index Notation

1. Express each of the following numbers in the common notation: 10^{10}, 10^{-8}, 2.8×10^{5}, 5.6×10^{-6}, 9.27×10^{15}.

2. The distance to the nearest fixed star, α Centauri, is 4.3 light years, or approximately $4.3 \times 5.88 \times 10^{12}$ mi. Express this last distance in the common notation.

The Greek letters α, β, and γ are read "alpha," "beta," and "gamma."

In each of the following cases the number after the name shows the approximate distance to the star in light years. Express these distances in miles, using the common notation:

3. Polaris, 44.

4. β Orionis, 500.

5. γ Pegasi, 300.

6. α Tauri (Aldebaran), 28.2.

7. The wave lengths of the visible colors in a spectrum vary from 3.9×10^{-7} m. to 7.5×10^{-7} m. Express these two numbers as decimal fractions.

8. The formula $V = V_0(1 + 3663 \cdot 10^{-6}\, t)$ gives the volume V of a perfect gas under constant pressure, where V_0 represents the volume at 0° C. (centigrade), and t represents the number of degrees of change in temperature. Taking $V_0 = 1$ and $t = 5$, find the value of V and express it both in the index notation and as a decimal fraction.

9. The linear expansion l of platinum is determined by the formula $l = l_0(1 + \alpha t + \beta t^2)$, where l_0 is the length at 0° C., α and β are constants, and t is the number of degrees of change in temperature. Taking $l_0 = 15$, $\alpha = 0.88 \times 10^{-5}$, and $\beta = 10^{-9}$, find the value of l when $t = 40°$.

10. The expansion of mercury is in accordance with the law $V = V_0(1 + \alpha t + \beta t^2 + \gamma t^3)$, where V_0 is the volume at 0° C., α, β, and γ are constants, and t is the change in temperature. Taking $V_0 = 7$, $\alpha = 181792 \cdot 10^{-9}$, $\beta = 175 \cdot 10^{-10}$, and $\gamma = 35116 \cdot 10^{-15}$, find the value of V when $t = 10°$.

Exercises. Review

1. Representing the weight in pounds of the hammer of a pile driver by W, the distance in feet through which the hammer drops by h, and the average resistance in pounds encountered by the pile by R, the distance i in inches which the pile is driven at each blow is given by the formula $i = \frac{Wh}{R}$. From this derive formulas for W, h, and R.

2. A flywheel 5.4 ft. in diameter is revolving at 120 R. P. M. (revolutions per minute). Find to the nearest tenth the speed in feet per second of a point on the rim of the wheel.

3. A formula for the horsepower H of an automobile engine is $H = \frac{d^2n}{2.5}$, where d represents the diameter in inches of the cylinders and n represents the number of cylinders. Taking the values 3.0, 3.1, 3.2, $\cdots$, 4.0 for d on the x axis, draw a graph of the formula for a 4-cylinder engine.

4. On the axes used in Ex. 3 draw graphs of the formula for a 6-cylinder engine and for an 8-cylinder engine, taking the same values for d as before. From a study of the three graphs write a brief statement of the facts revealed.

5. The formula $D_0 = \pi^{-1}(N + 2)P_c$ is used in finding the outside diameter D_0 of a spur gear with N teeth and the circular pitch P_c. From this formula derive one for N.

6. In experiments upon steam engines it has been determined that the steam pressure wasted in overcoming friction between the piston and the cylinder walls is given by the formula $P = 18\,d^{-\frac{1}{2}}$, where P is the pressure in pounds per square inch and d is the diameter of the cylinder in inches. Taking the values 6, 12, 18, $\cdots$, 48 for d, draw a graph showing the variations of P for cylinders from 6 in. to 48 in. in diameter. From the graph determine the approximate values of P for cylinders 15 in., 22 in., and 40 in. in diameter.

CHAPTER II

THE BINOMIAL THEOREM AND SERIES

1. The Binomial Theorem

Powers of a Binomial. By actual multiplication it is easily proved that

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

and $$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

We might, if we wished, continue to prove similar identities for higher powers of $a+b$, but the work would soon become tedious.

The Binomial Theorem. Instead of this we may, for any positive integral power of the binomial $a+b$, use the following formula:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n,$$

where $2!$ (called *factorial 2*) means $1 \cdot 2$, and similarly $3!$ means $1 \cdot 2 \cdot 3$, and so on.

When $n = 3$, we may substitute 3 for n in the formula and have

$$(a+b)^3 = a^3 + 3a^2b + \frac{3 \cdot 2}{1 \cdot 2}ab^2 + \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3}b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3,$$

which agrees, as we should expect, with the result found above by actual multiplication.

The law which is expressed by the above formula is known as the *Binomial Theorem.*

Proof of the Binomial Theorem. In seeking to prove the Binomial Theorem for positive integral powers, let us assume for the moment that the theorem as stated by the formula on page 53 is true for any positive integral power; for example, the kth power. We thus have

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{3!}a^{k-3}b^3 + \cdots + b^k. \quad (1)$$

If we multiply both members of equation (1) by $a+b$, we then have

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^k b + \frac{(k+1)k}{2!}a^{k-1}b^2 + \frac{(k+1)k(k-1)}{3!}a^{k-2}b^3 + \cdots + b^{k+1}. \quad (2)$$

It is evident that equation (2) is exactly what we shall have if we expand $(a+b)^{k+1}$ by the formula. We therefore see that if the theorem is true for the kth power, it is also true when $n = k+1$, because we have proved this by multiplying both members of (1) by $a+b$.

On page 53 we showed by actual multiplication and by substitution that the theorem is true when $n = 3$. Hence it follows from the above proof that the theorem is true when $n = 3+1$, or 4. Since the theorem is true when $n = 4$, it is therefore true when $n = 4+1$, or 5, and so on indefinitely. That is, the theorem is true for $(a+b)^n$ where n is any positive integer.

The theorem is also true for negative and fractional powers, but the general proof for these cases is beyond the scope of our present work.

This method of proving by progressive steps that a statement is true for any one of a group of cases (in this instance, for positive integral powers) is called *mathematical induction.*

This method of proof is sometimes called *complete induction.*

Formula for Any Term. A large part of the mathematics of finance and investment depends upon the Binomial Theorem, and in this connection a frequent problem is to find some particular term. From the formula on page 53, we can easily see that the third and fourth terms of the expansion of $(a+b)^n$ are respectively

$$\frac{n(n-1)}{2!}a^{n-2}b^2 \text{ and } \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3.$$

If we now represent any term, for example, the kth term, by the symbol T_k, we can readily infer that the general formula for any term in the expansion is

$$T_k = \frac{n(n-1)(n-2)\cdots \text{to } k-1 \text{ factors}}{(k-1)!}a^{n-(k-1)}b^{k-1}.$$

Using the Formulas. The following examples show how the binomial formulas are used:

1. Expand $(a^2 - \frac{1}{2}b)^5$.

Using the formula given on page 53, we have

$$\begin{aligned}(a^2 - \tfrac{1}{2}b)^5 &= (a^2)^5 + 5(a^2)^4(-\tfrac{1}{2}b) + \frac{5\cdot 4}{1\cdot 2}(a^2)^3(-\tfrac{1}{2}b)^2 \\ &\quad + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}(a^2)^2(-\tfrac{1}{2}b)^3 \\ &\quad + \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}a^2(-\tfrac{1}{2}b)^4 + (-\tfrac{1}{2}b)^5 \\ &= a^{10} - \tfrac{5}{2}a^8b + \tfrac{5}{2}a^6b^2 - \tfrac{5}{4}a^4b^3 + \tfrac{5}{16}a^2b^4 - \tfrac{1}{32}b^5.\end{aligned}$$

2. Find the sixth term in the expansion of $(3 - \frac{1}{2}m^2)^9$.

Letting $k = 6$ in the formula at the top of the page, we have

$$T_6 = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot 3^{9-5}(-\tfrac{1}{2}m^2)^5 = -\frac{5103\,m^{10}}{16}.$$

3. What is the sum of the numerical coefficients in the expansion of $(3a + \frac{1}{2}b)^5$?

It is evident that the required sum of the numerical coefficients is the expansion of $(3 + \frac{1}{2})^5$, and hence we have

$$(3 + \tfrac{1}{2})^5 = (\tfrac{7}{2})^5 = 525\tfrac{7}{32}.$$

Exercises. Binomial Theorem

Expand each of the following expressions:

1. $(2x + y)^4$. **4.** $(1 + r)^4$. **7.** 1.02^4. **10.** 1.05^5.

2. $(x - 2y)^5$. **5.** $(a + 0.4)^4$. **8.** 1.06^3. **11.** 1.03^6.

3. $(\frac{1}{3}a + 1)^6$. **6.** $(1 + 0.5)^3$. **9.** 1.05^4. **12.** 1.045^7.

13. Write the second term in the expansion of $(a + b)^n$, and then write a rule for finding this term.

14. In Ex. 13 find the third term in the expansion.

15. Write a rule for finding any term in the expansion of $(a + b)^n$ from the preceding term.

The rule may begin as follows: Multiply the coefficient of the preceding term by the exponent of a in that term and divide · · · .

16. In computing compound interest on investments it is frequently necessary to find a value like that of 1.04^5 to a specified number of decimal places. Find to the nearest 0.00001 the value of 1.04^5; of 1.06^6.

In the first case, we have from the formula

$$(1 + 0.04)^5 = 1 + 5 \times 0.04 + 10 \times 0.04^2 + 10 \times 0.04^3 + 5 \times 0.04^4 + 0.04^5$$
$$= 1 + 0.20 + 0.016 + 0.00064 + 0.0000128 + 0.0000001024$$
$$= 1.21665, \text{ to the nearest } 0.00001,$$

the last term of the expansion in this case being unnecessary.

In each of the following cases find the terms specified:

17. $(3a^2 - \frac{1}{2})^5$; 3d, 5th. **19.** $(4x^3 - \frac{3}{4})^6$; 4th, 5th.

18. $(1 + 0.04\frac{1}{4})^6$; 4th, 5th. **20.** 1.055^8; 5th, 7th.

21. The formula $A = P(1 + r)^n$ gives the amount A of an investment of P dollars, when the interest is compounded at the rate r for n periods. Find the amount at the end of 4 yr. of \$8500 invested at 5%, compounded semiannually.

In such a case we have eight periods, and the rate for each period is $2\frac{1}{2}\%$; that is, $n = 8$ and $r = 0.025$.

2. Arithmetic Series

Series. A succession of terms which proceed according to some fixed law is called a *series.*

The Binomial Theorem furnishes one example of a series, and another is found in the series of odd numbers, $1, 3, 5, 7, \cdots$.

A series which has a limited number of terms is called a *finite series,* and one in which the number of terms is unlimited is called an *infinite series.*

Arithmetic Series. A series in which each term after the first is formed by the addition of a constant quantity to the preceding term is called an *arithmetic series.*

For example, 5, 10, 15, 20 is an arithmetic series of four terms, and the series $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \cdots$ is an infinite arithmetic series.

An arithmetic series is also called an *arithmetic progression.*

A general expression for an arithmetic series is

$$a, a+d, a+2d, \cdots, l,$$

where d, which represents the constant quantity added each time, is called the *common difference.* In work with arithmetic series we find it convenient to designate the first term by a, the last term by l, the number of terms by n, and the sum of the terms by s. We often speak of a, d, n, l, and s as the five *elements* of a series.

Formula for Any Term. In the above series we see that the second term is $a+1\cdot d$, the third is $a+2\cdot d$, the fourth would be $a+3\cdot d$, and the nth term would therefore be $a+(n-1)d$.

Calling this nth term the last one (the last one in which we are at present interested), we then have

$$l=a+(n-1)d.$$

For example, the 8th term in the series $1, 7, 13, \cdots$ is $1+(8-1)\cdot 6$, or 43, and the 100th term in this series is $1+(100-1)\cdot 6$, or 595.

Formula for the Sum of n Terms. To find a general formula for the sum s of n terms of an arithmetic series we may write the series both forward and backward as follows:

$$s = a + (a + d) + \cdots + (l - d) + l$$

and

$$s = l + (l - d) + \cdots + (a + d) + a.$$

Adding these two expressions of n terms each, the result, which also contains n terms, is

$$2s = (a + l) + (a + l) + \cdots + (a + l) + (a + l),$$

or

$$2s = n(a + l).$$

The general formula for the sum s is therefore

$$s = \tfrac{1}{2}n(a + l).$$

Using the Formulas. The following examples illustrate the use of the formulas which have been developed.

1. Find the 20th term in the series 3, 5, 7, $\cdots$.

Since in this case $a = 3$, $d = 2$, and $n = 20$, we have

$$l = a + (n - 1)d = 3 + (20 - 1) \cdot 2 = 41.$$

2. Find the sum of the 20 terms in the series of Ex. 1.

Since we have found in Ex. 1 that $l = 41$, we have

$$s = \tfrac{1}{2}n(a + l) = \tfrac{1}{2} \cdot 20 \cdot (3 + 41) = 10 \cdot 44 = 440.$$

3. Derive a formula for the first term a of an arithmetic series in terms of the last term l, the common difference d, and the sum s.

From

$$s = \tfrac{1}{2}n(a + l),$$

we have

$$n = \frac{2s}{a + l}.$$

Substituting this value of n in the formula

$$l = a + (n - 1)d,$$

we have

$$l = a + \left(\frac{2s}{a + l} - 1\right)d,$$

or

$$a^2 - ad = l^2 + ld - 2ds.$$

Solving this quadratic equation for a, we then have

$$a = \tfrac{1}{2}[d \pm \sqrt{(2l + d)^2 - 8ds}].$$

Exercises. Arithmetic Series

In each of the following series find the term specified:

1. 2, 5, 8, · · · ; 7th.
2. 5, 16, 27, · · · ; 8th.
3. 3, 9, 15, · · · ; 9th.
4. $2\frac{1}{2}$, $3\frac{7}{8}$, $5\frac{1}{4}$, · · · ; 6th.
5. 0.5, 1.3, 2.1, · · · ; 9th.
6. 5, − 2, − 9, · · · ; 8th.
7. $1\frac{1}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, · · · ; 10th.
8. 23, 17, 11, · · · ; 12th.
9. − 7, 0, 7, · · · ; 20th.
10. a, $\frac{7}{8}a$, $\frac{3}{4}a$, · · · ; 10th.

In each of the following series find the sum of the number of terms specified:

11. 2, 9, 16, · · · ; 10.
12. 3, $4\frac{1}{2}$, 6, · · · ; 12.
13. $\frac{7}{8}$, $1\frac{3}{4}$, $2\frac{5}{8}$, · · · ; 10.
14. 0.3, 1, 1.7, · · · ; 15.
15. a, $\frac{7}{8}a$, $\frac{3}{4}a$, · · · ; 8.
16. 0, 4, 8, · · · ; 9.
17. 9, 6, 3, · · · ; 7.
18. − 9, − 6, − 3, · · · ; 7.
19. 3, $2\frac{1}{3}$, $1\frac{2}{3}$, · · · ; n.
20. a, $2a$, $3a$, · · · ; n.

21. Find a formula for a in terms of l, n, and s; a formula for d in terms of l, n, and s.

In each of the following cases find formulas for the other two elements of an arithmetic series in terms of

22. a, n, and s.
23. d, n, and s.
24. a, d, and l.
25. d, l, and s.
26. a, d, and s.
27. a, n, and l.

28. The first and last terms in an arithmetic series of five terms are 1 and 9. Find the other three terms.

The terms between any two terms of an arithmetic series are often called *arithmetic means*, but the name is not important and we shall not need to use it in our work.

29. Write the arithmetic series in which the first term is − 2 and the seventh term is 40.

30. Find the sum of twelve terms of an arithmetic series in which the first term is − 2 and the common difference is − 4.

Copy the following table and insert the missing numbers:

	a	l	n	d	s
31.	3	23		4	
32.	$6\frac{1}{2}$	128	82		
33.	3	21			120
34.	1144			-572	0
35.		-1382	7	-691	
36.	1270	15,342			182,732
37.	32,000	0	16,001		
38.	28,500		3	$-18,500$	
39.	1.72		10	3.88	
40.		77		5	623

41. Find the 25th odd integer; the 100th even integer.

42. What is the smallest number of terms in the series 5, 8, 11, $\cdots$ that can be taken in order that their sum shall exceed 500?

Finding a formula for s in terms of a, d, and n, we have $a = 5$, $d = 3$, and $s = 500$. Substituting and solving for n, we then determine the appropriate integral value for n.

43. A teacher who has reached the maximum salary of \$2880 a year in a certain city is offered another position at \$2400 the first year with an annual increase of \$150 for 10 yr. Find the total difference in the salaries in the two positions during a period of 3 yr.; of 5 yr.; of 10 yr.

44. Write the arithmetic series in which $a = 3$, $l = -17$, and $s = -77$.

45. In an arithmetic series in which the first term is one third of the second prove that the sum of the terms is n^2 times the first term.

3. Geometric Series

Geometric Series. A series in which each succeeding term after the first is formed by multiplying the preceding term by a constant quantity is called a *geometric series.*

An example of a geometric series, which is also called a *geometric progression,* is 2, 4, 8, 16.

A general expression for a geometric series is

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ l,$$

where r, which represents the constant multiplier, is called the *ratio.* The other elements of the series are the first term (a), the last term (l), the number of terms (n), and the sum of the terms (s).

Formula for Any Term. In the above series the first term is a, the second is ar, the third is ar^2, and so on. The nth term is therefore ar^{n-1}. Calling this nth term the last one, we thus have the general formula

$$l = ar^{n-1}.$$

Formula for the Sum of n Terms. Writing the sum of a geometric series as

$$s = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \qquad (1)$$

we may multiply both sides by r and have

$$rs = ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n. \qquad (2)$$

Subtracting (1) from (2), we then have

$$rs - s = -a + ar^n;$$

from which we have the general formula

$$s = \frac{ar^n - a}{r - 1}.$$

Since $ar^n = lr$, we may also write this formula as

$$s = \frac{lr - a}{r - 1}.$$

Exercises. Geometric Series

1. In the series 3, 21, 147, $\cdots$ find the fifth term; the seventh term.

For the fifth term, $a = 3$, $r = 7$, and $n = 5$, and hence we have

$$l = ar^{n-1} = 3 \cdot 7^4 = 3 \cdot 2401 = 7203.$$

2. In Ex. 1 find the sum of four terms; of six terms.

For four terms, $a = 3$, $r = 7$, and $n = 4$, and hence we have

$$s = \frac{ar^n - a}{r - 1} = \frac{3 \cdot 7^4 - 3}{7 - 1} = \frac{7200}{6} = 1200.$$

3. Find the tenth term in the series 3, 6, 12, $\cdots$.

4. Find the sum of ten terms in the series 4, 20, 100, $\cdots$.

5. Find the second term of the geometric series in which the first and third terms are 7 and 343.

6. Solve Ex. 5 when the first term is m and the third is n.

7. Find the second and third terms of a geometric series in which the first term is 3 and the fourth is 81.

8. Find the value of r in the series 3, $3r$, $3r^2$, 24.

9. Find a formula for a in terms of l, r, and s.

10. Find a formula for r in terms of a, l, and s.

11. Find a formula for l in terms of n, r, and s.

12. Show that, if a, b, and c form a geometric series, $\frac{a}{c} = \frac{a^2}{b^2}$.

13. If $3(x + 4)$, x, and 3 form a geometric series, what is the value (or values) of x?

14. If the first two terms of a geometric series are p and q, what is the sixth term?

15. Find the sum of six terms of the series 1.04, 1.04^2, 1.04^3, $\cdots$.

16. Find the sum of twelve terms of the geometric series in which the first two terms are n and nk.

Infinitely Decreasing Geometric Series. When r is a positive proper fraction, the successive terms of a geometric series constantly decrease. Thus, if we take n sufficiently large, we can make the nth term as small as we wish; that is, we can make it as near zero as we wish.

For example, in the series 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\cdots$, the terms are steadily decreasing, and if we take $n = 100$, the nth term will be exceedingly small.

Since $ar^n < a$ and $r < 1$, we may avoid negative numbers by writing the formula for the sum s of n terms first as

$$s = \frac{a - ar^n}{1 - r},$$

and then as

$$s = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

Now by taking n large enough, we can make ar^n, and hence $\frac{ar^n}{1 - r}$, approach as near as we wish to zero. We thus see that s approaches as near as we wish to $\frac{a}{1 - r}$, or, as we say, this fraction is the *limit* which s approaches as n increases indefinitely. For brevity, we say that when r is a positive proper fraction and n increases indefinitely

$$s = \frac{a}{1 - r}.$$

Repeating Decimals. A decimal like $0.5242424\cdots$, in which a certain group of figures is continually repeated, is called a *repeating decimal.*

It is evident that the decimal $0.5242424\cdots$ may be written as

$$0.5 + 0.024 + 0.00024 + \cdots,$$

where 0.024, 0.00024, $\cdots$ form an infinitely decreasing geometric series in which $a = 0.024$ and $r = 0.01$. We therefore have

$$s = \frac{a}{1 - r} = \frac{0.024}{1 - 0.01} = \frac{0.024}{0.99} = \frac{24}{990}.$$

Hence to express the decimal as a common fraction we have

$$0.5242424\cdots = 0.5 + \tfrac{24}{990} = \tfrac{5}{10} + \tfrac{24}{990} = \tfrac{519}{990} = \tfrac{173}{330}.$$

Exercises. Review of Series

Find the sum of each of the following infinite series:

1. $20, 5, 1\frac{1}{4}, \cdots$. 3. $1, \frac{1}{2}, \frac{1}{4}, \cdots$. 5. $4, -2, 1, -\frac{1}{2}, \cdots$.

2. $30, 3, 0.3, \cdots$. 4. $1, \frac{1}{3}, \frac{1}{9}, \cdots$. 6. $5, -1, \frac{1}{5}, -\frac{1}{25}, \cdots$.

In such cases it is understood that the word "sum" is used to mean the limit of the sum.

Express the following decimals as common fractions:

7. $0.333\cdots$. 9. $0.232323\cdots$. 11. $0.1666\cdots$.

8. $0.777\cdots$. 10. $0.2325325\cdots$. 12. $0.34125125\cdots$.

13. The first term of an arithmetic series is -15 and the sixth term is 45. Write the first six terms.

14. The first term of an arithmetic series is 3 and the ninth term is 51. What is the tenth term?

15. Find the sum of ten terms of the arithmetic series in which the second term is 7 and the fourth term is 19.

16. A body falling freely travels 16 ft. the first second, and in each succeeding second it travels 32 ft. more than in the second just preceding. If a stone dropped from a balloon reaches the ground in 12 sec., how far does it fall in the last second and how high is the balloon?

The results will be only approximate since 16 ft. and 32 ft. are correct only to the nearest 1 ft., and air resistance is disregarded.

17. Some countries use 24-hour time, counting midnight as 24 o'clock. If a clock strikes all the hours and also strikes once every quarter of an hour, how many strokes does it make during a day?

18. If the first term of a geometric series is 5 and the ratio is 2, which term is 1280?

19. If you take half of a line that is 8 in. long, half of what is left, and so on, what is the sum of all these parts?

CHAPTER III

LOGARITHMS

1. General Principles

Purpose of Logarithms. One of the most important of the many devices to simplify calculation depends upon the use of logarithms, which were invented by John Napier, a Scotchman, early in the seventeenth century. By this method multiplication is reduced to addition, division to subtraction, finding powers to multiplication, and finding roots to division. By means of logarithms it is about as easy to find the value of $\sqrt[12]{100}$ as to find that of $\sqrt[3]{100}$, and either operation is a very simple one.

Logarithm. We may, if we wish, consider all positive numbers as powers of some particular positive number, which is called the *base*; that is, we may write

$$N = b^l,$$

where N is the number which we regard as the lth power of the base b. The exponent l, which indicates the power, is called the *logarithm* of N to the base b. Other bases may be taken, but in our work at present the base 10 will be used.

As we have already seen in studying exponents on page 50,

$$10{,}000 = 10^4,$$

and hence the logarithm of 10,000 is 4, which we abbreviate by writing

$$\log 10{,}000 = 4.$$

In the same way we have the following relations:

$$10 = 10^1 \quad \text{and} \quad \log 10 = 1;$$
$$10^0 = 1 \quad \text{and} \quad \log 1 = 0;$$
$$0.1 = \tfrac{1}{10} = 10^{-1} \quad \text{and} \quad \log 0.1 = -1.$$

Logarithms in Multiplication and Division. Considering only logarithms to the base 10, we may write

$$a = 10^l \quad \text{and} \quad b = 10^{l'},$$

where a and b are two numbers and l and l' are their respective logarithms; that is,

$$l = \log a \quad \text{and} \quad l' = \log b.$$

Then $$ab = 10^l \cdot 10^{l'} = 10^{l+l'},$$

and hence, by the definition of logarithm, we have

$$l + l' = \log ab,$$

or $$\log ab = \log a + \log b.$$

That is, expressed as a general law,

The logarithm of the product of two numbers is the sum of the logarithms of the numbers.

This law and those which follow are easily proved for any base.

For the two numbers a and b, above, we also have

$$\frac{a}{b} = \frac{10^l}{10^{l'}} = 10^{l-l'};$$

whence $$l - l' = \log \frac{a}{b},$$

or $$\log \frac{a}{b} = \log a - \log b.$$

That is, expressed as a general law,

The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.

Similarly, for the nth power of a, we have

$$a^n = (10^l)^n = 10^{nl};$$

whence $$nl = \log a^n,$$

or $$\log a^n = n \log a.$$

That is, expressed as a general law,

The logarithm of a power of a number is the logarithm of the number multiplied by the exponent.

Exercises. General Principles

1. What is meant by the expression $10^{\frac{1}{2}}$? What, to the nearest 0.001, is the value of $10^{\frac{1}{2}}$? Of what number is 0.5 the logarithm?

2. If we know that the value of $10^{\frac{3}{4}}$, or $10^{0.75}$, is 5.624, what is the logarithm of 5.624?

3. What is $\log \frac{1}{100}$, or $\log 0.01$? $\log 0.001$? $\log 0.0001$?

4. Since $\log 10 = 1$ and $\log 100 = 2$, between what two integers does $\log 85$ lie?

Write the integers between which lie the logarithms of

5. 86. **7.** 325. **9.** 6339. **11.** 82.780.

6. 86.7. **8.** 325.9. **10.** 8275. **12.** 8.2780.

Show that the following statements are true:

13. $\log 1 + 3 \log 10 + \log 100 + \log 1000 + \log 0.001 = 5$.

14. $9 \log 1 + 8 \log 10{,}000 + \log 0.1 + \log 0.000001 = 25$.

15. If we take $\log 35$ as 1.5441 and $\log 53.7$ as 1.7300, what does this tell us concerning the power of 10 that is equal to 35 and the one that is equal to 53.7?

16. In Ex. 15 find the logarithm of 35×53.7; of $53.7 \div 35$; of 35^3; of 53.7^8.

17. Taking $\log 71.6$ as 1.8549, $\log 2.58$ as 0.4116, $\log 454$ as 2.6571, $\log 9.55$ as 0.9800, and $\log 15$ as 1.1761, find $\log (71.6 \times 2.58)$; $\log (454 \div 9.55)$; $\log (15 \times 71.6)^2$.

18. In Ex. 17 find $\log (71.6 \times 2.58 \times 454 \times 9.55 \times 15)$.

Just as $\log ab = \log a + \log b$, so $\log abc = \log a + \log b + \log c$, and so on for any number of factors.

19. In Ex. 17 find $\log \dfrac{9.55 \times 15 \times 454}{71.6 \times 2.58}$.

We now see that if we can find the number that corresponds to a given logarithm, the operations indicated in Exs. 16–19 will be simplified.

Logarithm of a Root. For fractional exponents, the formula for the logarithm of a power stated on page 66 becomes

$$\log a^{\frac{1}{n}} = \frac{1}{n} \log a.$$

We therefore have the following general law:

The logarithm of a root of a number is the logarithm of the number divided by the index of the root.

The truth of this law may be shown, as on page 66, by taking

$$a = 10^l, \quad \text{or} \quad l = \log a.$$

Then $$a^{\frac{1}{n}} = 10^{\frac{l}{n}};$$

whence $$\frac{l}{n} = \log a^{\frac{1}{n}},$$

or $$\log a^{\frac{1}{n}} = \frac{1}{n} \log a.$$

Characteristic and Mantissa. A logarithm usually consists of an integer plus a decimal fraction. The integral part is called the *characteristic*, and the decimal part the *mantissa*.

For example, in 1.5441, which was given on page 67 as the logarithm of 35, the characteristic is 1 and the mantissa is .5441.

While the better form of writing such a decimal fraction is 0.5441, in referring to a mantissa we shall use the form .5441 to avoid possible confusion with a logarithm having the characteristic 0.

From the following table of different powers of 10,

$10^0 = 1$	$10^2 = 100$	$10^{-1} = 0.1$	$10^{-3} = 0.001$
$10^1 = 10$	$10^3 = 1000$	$10^{-2} = .01$	$10^{-4} = .0001$

it is easily seen that the logarithm of a number between 1 and 10 is 0 plus some fraction; between 10 and 100 it is 1 plus some fraction, and so on. Similarly, the logarithm of a number between 0.1 and 0.01 lies between -1 and -2, and hence is -2 plus some fraction; between 0.01 and 0.001 it is -3 plus some fraction, and so on.

For convenience the mantissa is always taken as positive, but the characteristic may be either positive or negative.

Finding the Characteristic. From the discussion on page 68 we have the following summary:

1. *The characteristic of the logarithm of a number greater than 1 is positive, and is one less than the number of integral places in the number.*

For example, $\log 86 = 1 + \text{some mantissa}$,
and $\log 127.8 = 2 + \text{some mantissa}$.

2. *The characteristic of the logarithm of a number between 0 and 1 is negative, and is numerically one more than the number of zeros immediately following the decimal point.*

For example, $\log 0.04 = -2 + \text{some mantissa}$,
and $\log 0.00083 = -4 + \text{some mantissa}$.

Negative Characteristic. If we take the mantissa of log 0.02 as .3010, we thus see that

$$\log 0.02 = -2 + .3010 = 0.3010 - 2.$$

We cannot write this as -2.3010, because the mantissa is positive, and so we usually write it as $\bar{2}.3010$ to indicate that only the 2 is negative.

We may also write $\log 0.02 = 8.3010 - 10$, $\log 0.02 = 1.3010 - 3$, or any similar form that will indicate the negative characteristic 2.

Mantissa and Decimal Point. We see that the above logarithm of 0.02 means that

$$10^{0.3010-2} = 0.02.$$

Multiplying both members in turn by 10, 10^2, and 10^3, we have

$10^{0.3010-1} = 0.2$, or $\log 0.2 = \bar{1}.3010$;
$10^{0.3010} = 2$, or $\log 2 = 0.3010$;
and $10^{0.3010+1} = 20$, or $\log 20 = 1.3010$.

We thus see that the mantissas of log 20, log 2, log 0.2, and log 0.02 are all the same; that is,

Numbers which differ only in the position of the decimal point have the same mantissas for their logarithms.

Exercises. Characteristic and Mantissa

1. In finding log 79 we first determine the characteristic. State the characteristic and your reasons for choosing the number that you give.

2. If the characteristic of a logarithm is 3, how many integral places are there in the number that corresponds to this logarithm? State the reasons for your answer.

3. How many integral places are there in the number which has the logarithm 4.8296? the logarithm 8.8296? State the reasons for your answers.

Write the characteristics of the logarithms of

4. 64.	**12.** 8235.	**20.** 0.7.	**28.** 0.0004.
5. 64.8.	**13.** 823.5.	**21.** 0.07.	**29.** 0.0044.
6. 648.	**14.** 82.35.	**22.** 0.78.	**30.** 0.0444.
7. 6.48.	**15.** 8.235.	**23.** 0.786.	**31.** 0.4444.
8. 6480.	**16.** 82,350.	**24.** 0.006.	**32.** 0.0404.
9. 6485.	**17.** 0.8235.	**25.** 0.0061.	**33.** 1.0404.
10. 6485.7.	**18.** 0.08235.	**26.** 0.0006.	**34.** 2.0404.
11. 64,857.	**19.** 0.008235.	**27.** 0.00061.	**35.** 10.0044.

Taking log 645 as 2.8096, find

36. log 64.5.	**39.** log 6450.	**42.** log 645,000.
37. log 6.45.	**40.** log 64,500.	**43.** log 0.00645.
38. log 0.645.	**41.** log 0.0645.	**44.** log 6,450,000.

Taking log 36,630 as 4.5639 and log 8.154 as 0.9114, find

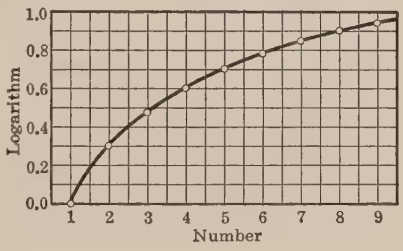
45. log 3.663.	**49.** log 0.3663^5.	**53.** log $3.663^{\frac{1}{3}}$.
46. log 36.63.	**50.** log 81.54^{12}.	**54.** log $\sqrt{815.4}$.
47. log 815.4.	**51.** log (36.63×8.154).	**55.** log $36.63^{\frac{2}{3}}$.
48. log 0.08154.	**52.** log $(366.3 \div 81.54)$.	**56.** log $8.154^{\frac{1}{7}}$.

Accuracy of Logarithms. Just as most roots of numbers are only approximate, so logarithms are in general only approximate; that is, they are accurate only to a certain number of decimal places. The logarithms of numbers are computed by methods of higher mathematics and are then arranged in convenient tables. These tables are designated as four-place tables (that is, tables which give the logarithms to the nearest 0.0001), five-place tables, six-place tables, and so on, according to the degree of accuracy to which the logarithms have been computed.

For example, from a four-place table we can find that

$\log 1 = 0.0000$	$\log 4 = 0.6021$	$\log 7 = 0.8451$
$\log 2 = .3010$	$\log 5 = .6990$	$\log 8 = .9031$
$\log 3 = .4771$	$\log 6 = .7782$	$\log 9 = .9542$

Let us now consider the accompanying graph, which is drawn by locating the points that correspond to the above logarithms. While the graph is a curve, for practical purposes we may regard a very short distance on it, such as the distance corresponding to the difference between 3.14 and 3.15, as a straight line. Hence the logarithm of an intermediate number like 3.145 will correspond to the point on the graph halfway between 3.14 and 3.15.



We could not, of course, find log 2 in this way, because half the difference between log 3 and log 1 is $\frac{1}{2}(0.4771 - 0.0000)$, or 0.2386, approximately. As shown above, $\log 2 = 0.3010$.

At 8.5, however, the curve has flattened out more, and we see that half the difference between log 8 and log 9 is about 0.0256; that is, log 8.5 is about $0.9031 + 0.0256$, or 0.9287, which is only 0.0007 less than the value 0.9294 given by a four-place table. Hence in finding log 3.145 the method will evidently be sufficiently accurate.

Interpolation. The method suggested on page 71, which is known as *interpolation*, may be applied in finding the logarithm of a number that is not given in a table, as shown in the following examples:

1. Taking log 2.04 as 0.3096 and log 2.05 as 0.3118, find by interpolation log 2.045.

Since 2.045 is halfway between 2.04 and 2.05, the logarithm is approximately halfway between 0.3096 and 0.3118; that is,

$$\begin{aligned} \log 2.05 &= 0.3118 \\ \log 2.04 &= \underline{.3096} \\ \text{Difference} &= 0.0022 \end{aligned}$$

Taking $\frac{1}{2}$ of this difference, we have 0.0011, and hence

$$\log 2.045 = 0.3096 + 0.0011 = 0.3107.$$

2. Taking log 45.2 as 1.6551 and log 45.3 as 1.6561, find by interpolation log 45.22.

Since 45.22 is 0.2 of the way from 45.2 to 45.3, we must add 0.2 of the difference between 0.6561 and 0.6551 to log 45.2. Proceeding as in Ex. 1, above, we have

$$\log 45.22 = 1.6551 + 0.0002 = 1.6553.$$

Exercises. Interpolation

Taking log 3.08 as 0.4886, log 3.09 as 0.4900, log 48.0 as 1.6812, log 48.1 as 1.6821, log 733 as 2.8651, and log 734 as 2.8657, find the logarithms of the following numbers:

1. 3.085.	**4.** 48.05.	**7.** 733.1.	**10.** 3.083.
2. 3.082.	**5.** 48.03.	**8.** 733.2.	**11.** 48.07.
3. 3.087.	**6.** 48.08.	**9.** 733.4.	**12.** 733.9.

Using the logarithms given for Exs. 1–12, find

13. $\log 3.083^2$.	**17.** $\log 480.6^4$.
14. $\log \sqrt{30.84}$.	**18.** $\log 48.03^{\frac{1}{3}}$.
15. $\log (3.086 \times 48.07)$.	**19.** $\log (30.84 \div 4.802)$.
16. $\log (4.809 \times 73.33)$.	**20.** $\log (733.5 \div 30.88)$.

2. Computations with Logarithms

Finding a Logarithm. On pages 358 and 359 is given a four-place table of the logarithms of numbers from 1.00 to 9.99, except that the characteristic, which is always zero for any number that has only one integral place (page 69), is omitted. The use of the table will be evident from the following illustrative examples:

1. Find the logarithm of 1.54.

On page 358 we first look in the column headed **N** for 1.5, and then following along the horizontal line **1.5** to column 4, we find .1875; that is,

$$\log 1.54 = 0.1875.$$

2. Find the logarithm of 1.547.

Since 1.547 is 0.7 of the way from 1.54 to 1.55, we may use interpolation, as described on page 72; that is, finding log 1.54 and log 1.55 as in Ex. 1, above, we have

$$\begin{aligned} \log 1.55 &= 0.1903 \\ \log 1.54 &= \underline{.1875} \\ \text{Difference} &= 0.0028 \end{aligned}$$

Taking 0.7 of this difference, we have 0.0020, approximately, and adding this to log 1.54, we have

$$\log 1.547 = 0.1875 + 0.0020 = 0.1895.$$

Instead of computing the number to be added, we may use the *difference columns* given at the right of each page of the table. These columns, which are marked by the figures **1–9** in smaller type, show the number to be added in each case.

For example, after finding that log 1.54 = 0.1875, as above, we simply follow along the horizontal line **1.5** to the difference column **7**. We there find 20, which means that 0.0020 is the number to be added to log 1.54. Adding this mentally, we thus find directly that

$$\log 1.547 = 0.1895.$$

This method is usually more convenient than the longer method of interpolation, and although there is sometimes an error of 1 or 2 in the fourth decimal place, it is sufficiently accurate for most practical cases. Since some tables do not have difference columns, however, it is desirable that students be able to find any logarithm readily by the method of interpolation described on page 72.

3. Find the logarithm of 300.

Since $300 = 10^2 \times 3,$

we see that $\log 300 = 2 + \log 3.$

Since $3 = 3.00$, we therefore look up log 3 in the table in horizontal line **3.0** under column **0**, and find it to be 0.4771. Hence

$$\log 300 = 2 + 0.4771 = 2.4771.$$

In practice we simply apply the rules for the characteristic given on page 69; that is, we see at once that the characteristic of log 300 is 2, and we write the 2 first. We then look up the mantissa of log 3 in the table and have

$$\log 300 = 2.4771.$$

4. Find the logarithm of 0.0006178.

In this case the characteristic is -4, and so we first write $\bar{4}$. Since the mantissa of log 0.0006178 is the same as that of log 6.178, we then look up the mantissa of log 6.178 in the table in the usual way (using the difference column **8**) and find it to be 0.7909. Hence

$$\log 0.0006178 = \bar{4}.7909.$$

As already stated, we shall often find one of the equivalent forms, $0.7909 - 4$, $6.7909 - 10$, $16.7909 - 20$, and so on, more convenient for practical work.

Exercises. Finding Logarithms

Using the table on pages 358 and 359, find the logarithm of

1. 52.	**11.** 6.436.	**21.** 9364.	**31.** 6000.
2. 520.	**12.** 75.82.	**22.** 0.936.	**32.** 6100.
3. 520.5.	**13.** 69.31.	**23.** 0.639.	**33.** 6125.
4. 700.	**14.** 11.12.	**24.** 82.75.	**34.** 0.612.
5. 690.	**15.** 121.5.	**25.** 3.142.	**35.** 0.0615.
6. 692.	**16.** 0.0758.	**26.** 0.1832.	**36.** 0.000786.
7. 692.5.	**17.** 0.7586.	**27.** 12,750.	**37.** 0.09999.
8. 6928.	**18.** 0.00785.	**28.** 172,800.	**38.** 1,250,000.
9. 6982.	**19.** 0.7085.	**29.** 17,820.	**39.** 2,500,000.
10. 6.985.	**20.** 0.07805.	**30.** 170,200.	**40.** 0.000250.

Antilogarithm. The number which corresponds to a given logarithm is called an *antilogarithm*, which is usually abbreviated by writing "antilog."

For example, log 593 = 2.7731, and hence the antilogarithm of 2.7731 is 593, which we may write as antilog 2.7731 = 593.

Finding an Antilogarithm. An antilogarithm is found by first locating the position of the given logarithm in the table. We can then find the antilogarithm by simply reversing the methods of finding the logarithm of a number, as shown in the following examples:

1. Find the antilogarithm of 2.5944.

Disregarding the characteristic 2 at first, we try to locate the position of .5944 in the table. Running down the column headed 0 on page 358, we find .5911 in horizontal line 3.9 and .6021 in line 4.0. We thus see that .5944 is probably in line 3.9, and following along this line we find it in column 3; that is, .5944 is the mantissa of log 3.93, of log 39.3, of log 393, or of any other number having the figures 3, 9, and 3 in this order. The characteristic 2, however, shows that the antilogarithm has three integral places, and we therefore find that

$$\text{antilog } 2.5944 = 393.$$

2. Find the antilogarithm of 1.3931.

Proceeding as in Ex. 1, we locate the position of .3931 as lying between .3927 and .3945, the mantissas of log 2.47 and log 2.48 respectively. Reversing the process of interpolation given on page 72, we see that the difference between .3931 and .3927 is .0004, and the total difference between .3945 and .3927 is .0018. Hence the first difference is $\frac{4}{18}$, or 0.2 approximately, of the total difference; that is, the difference .0004 corresponds to a fourth figure 2 in the antilogarithm. Thus, since the characteristic 1 shows that there are two integral places in the required antilogarithm, we see that

$$\text{antilog } 1.3931 = 24.72.$$

Instead of this, if we notice first that .3931 is .0004 greater than .3927 (the mantissa of log 2.47) and then look in the difference columns on line 2.4, we shall find that 4 is given as the difference in column 2; that is, we can find at once that

$$\text{antilog } 1.3931 = 24.72.$$

3. Find the antilogarithm of $\bar{3}.9793$.

Proceeding as before, we locate the position of .9793 as coming after .9791, which is the mantissa of log 9.53. Then since .9793 is .0002 larger than .9791, we look for 2 in the difference columns in line 9.5, and find it given in both columns 4 and 5.

To determine whether to use 4 or 5 we must resort to interpolation. In this case we see that the total difference between .9791 and .9795 (the next larger mantissa) is .0004, and hence our difference of .0002 is $\frac{1}{2}$, or 0.5, of this total difference; that is, the fourth figure of the antilogarithm is 5 (not 4). Since the characteristic is -3, there are two zeros immediately following the decimal point; that is, we have

$$\text{antilog } \bar{3}.9793 = 0.009535.$$

We thus see that although we can often avoid the longer process of interpolation, it is necessary in certain cases to know how to interpolate even if we have tables that are supplied with difference columns.

Exercises. Antilogarithms and Logarithms

Using the table on pages 358 and 359, find the antilogarithm of

1. 4.9977.	**6.** 2.9846.	**11.** $\bar{1}.8280$.	**16.** $4.7409 - 10$.
2. 2.8785.	**7.** $\bar{2}.8747$.	**12.** $\bar{2}.6675$.	**17.** $0.8300 - 2$.
3. $\bar{2}.8380$.	**8.** 3.6852.	**13.** $\bar{4}.8306$.	**18.** $8.8303 - 10$.
4. $\bar{3}.9546$.	**9.** 3.2620.	**14.** 2.7793.	**19.** $3.9296 - 4$.
5. 3.7845.	**10.** 3.5210.	**15.** $\bar{1}.8817$.	**20.** $14.8045 - 20$.

21. If the logarithm of the product of two numbers is 2.8211, what is the product of the numbers?

22. Find log 0.00386 and divide it by 2; by 4; by 5.

From the table $\log 0.00386 = \bar{3}.5866$. Since this cannot be divided by 2 in either this form or the form $0.5866 - 3$, on account of the negative characteristic 3, we must use one of the equivalent forms $1.5866 - 4$, $7.5866 - 10$, or some similar form that will give a negative part at the right that is divisible by 2. In this case any one of these forms gives a result of $0.7933 - 2$, or $\bar{2}.7933$.

23. Find log 0.5, log 0.00752, and log 0.007284. Divide each by 3, and find the antilogarithm of each result.

Computing by Logarithms. In using logarithms to save time and labor in making computations we apply the laws given on page 66, as shown in the following typical examples:

1. Find the product of 2.75 and 37.6.

Since $\log ab = \log a + \log b,$

we first look up the logarithms of 2.75 and 37.6 and have

$$\log 2.75 = 0.4393$$
$$\log 37.6 = \underline{1.5752}$$

Adding, we have 2.0145

Then $\text{antilog } 2.0145 = 103.4.$

Multiplying in the usual way, we have $2.75 \times 37.6 = 103.4.$

2. Find the product of 8.27, 48.62, and $-0.118.$

Although we cannot find the logarithm of a negative number, we may find the product of 8.27, 48.62, and 0.118 and then prefix the minus sign, since we see that the result will be negative. We therefore have

$$\log 8.27 = 0.9175$$
$$\log 48.62 = 1.6868$$
$$\log 0.118 = \underline{0.0719 - 1}$$

Adding, we have $2.6762 - 1 = 1.6762$

Then $\text{antilog } 1.6762 = 47.44.$

The required product is therefore $-47.44.$ If we multiply in the usual way, the result is $-47.4463132,$ which to four figures is $-47.45.$

In practical work such a difference can often be disregarded, but it should be remembered that logarithms generally give only approximate results, the degree of accuracy depending upon the number of decimal places in the table used. With a four-place table we can give results only to four figures, and the last figure may be incorrect.

3. Find the quotient of $23.04 \div 0.256.$

Since $\log \frac{a}{b} = \log a - \log b,$

we have $\log 23.04 = 1.3624$

$$\log 0.256 = \underline{0.4082 - 1}$$

Subtracting, we have $0.9542 + 1 = 1.9542$

Then $\text{antilog } 1.9542 = 90.$

In this case also, actual division would give the same result.

4. Find the quotient of $78.32 \div 627.8$.

From the tables we find that $\log 78.32 = 1.8939$ and $\log 627.8 = 2.7979$. In order to keep the mantissa positive when subtracting a larger logarithm from a smaller one, we change the form of $\log 78.32$ and proceed as follows:

$$\log 78.32 = 3.8939 - 2$$
$$\log 627.8 = \underline{2.7979}$$

Subtracting, we have $\quad 1.0960 - 2 = \bar{1}.0960$

Then $\quad \text{antilog } \bar{1}.0960 = 0.1247.$

5. Find the value of 0.0256^3.

We have $\quad \log a^n = n \log a,$

and from the tables $\log 0.0256 = \bar{2}.4082$. In order to avoid confusion in multiplying by 3, however, we proceed as follows:

$$\log 0.0256 = 0.4082 - 2$$
$$\underline{\qquad\qquad 3}$$

Multiplying, we have $\quad 1.2246 - 6 = \bar{5}.2246$

Then $\quad \text{antilog } \bar{5}.2246 = 0.00001677.$

By actual multiplication the result is 0.000016777216.

6. Find the value of $\sqrt[4]{3.142}$.

We have $\quad \log a^n = n \log a,$

in which $n = \frac{1}{4}$ and $\log a = \log 3.142 = 0.4972$. We therefore have

$$4)\underline{0.4972}$$
$$0.1243$$

Then $\quad \text{antilog } 0.1243 = 1.331.$

7. Find the value of the expression $\sqrt[5]{0.0137 \times 2.624 \times 8.075}$.

We have
$$\log 0.0137 = \bar{2}.1367$$
$$\log 2.624 = 0.4190$$
$$\log 8.075 = \underline{0.9072}$$

Adding, we have $\quad \bar{1}.4629$

On account of the negative characteristic we then change the form of $\bar{1}.4629$ before dividing by 5, and have

$$5)\underline{4.4629 \ - 5}$$
$$0.89258 - 1 = \bar{1}.8926, \text{ approximately.}$$

Then $\quad \text{antilog } \bar{1}.8926 = 0.7809.$

Without logarithms such a computation would be very difficult.

Exercises. Computing by Logarithms

Perform the following computations by logarithms:

1. 2.66×38.4.
2. 5.57×429.
3. 20.7×30.2.
4. 409×207.
5. 67×4769.
6. 49×289.6.
7. 36×38.92.
8. 63×4.178.
9. 7×0.1728.
10. 6×0.0146.
11. $3.9 \div 6.8$.
12. $75.7 \div 8.3$.
13. $62.8 \div 0.76$.
14. $0.008 \div 0.89$.
15. $0.072 \div 0.06$.
16. $32.83 \div 0.7$.
17. $4.009 \div 9.9$.
18. $6 \div 3.142$.
19. $7 \div 31.47$.
20. $0.9 \div 3.14$.
21. $\sqrt{3}$.
22. $\sqrt[3]{3}$.
23. $\sqrt[4]{3}$.
24. $\sqrt[5]{9}$.
25. $\sqrt[6]{728}$.
26. $\sqrt[7]{347.6}$.
27. $\sqrt[8]{0.0004}$.
28. 62.37^{7}.
29. 3.107^{13}.
30. 0.46^{20}.
31. Find the value of $\sqrt{4.75 \times 32.95 \times 617.9}$.
32. Find the value of $\sqrt[3]{0.6 \times 0.0768 \times 328.4}$.
33. Find the value of $5.76 \times 29.35 \times 82.86 \times 0.09$.

Perform the following multiplications by logarithms:

34. 3.389×0.000828.
35. 49.78×0.000049.
36. 0.672×0.003235.
37. -89.8×67.63.
38. $-37.82 \times (-4.79)$.
39. $-9.678 \times (-0.0068)$.

Perform the following divisions by logarithms:

40. $37.78 \div 52.81$.
41. $0.3988 \div 4.427$.
42. $976.9 \div 0.007348$.
43. $0.08196 \div 87.99$.
44. $0.7398 \div 0.5929$.
45. $0.0006873 \div 0.4289$.
46. $\sqrt{62.56} \div \sqrt[3]{1.732}$.
47. $\sqrt[4]{3.825} \div \sqrt{87.85}$.

Perform the following operations by logarithms:

48. $32^{\frac{2}{5}}$.
49. $468^{\frac{4}{5}}$.
50. $54.92^{\frac{3}{4}}$.
51. $787.9^{\frac{5}{6}}$.
52. $9.142^{\frac{2}{5}}$.
53. $0.0032^{\frac{3}{8}}$.
54. $(-41)^{\frac{2}{3}}$.
55. $(-2.21)^{\frac{3}{5}}$.

Cologarithm. In a long and complicated expression like

$$\frac{3.246 \times 0.878 \times 12.44}{6.725}$$

it is evident that instead of dividing the numerator by 6.725 we may multiply it by $\frac{1}{6.725}$. Since multiplication by logarithms reduces to addition, we may then add the logarithm of the reciprocal of the divisor instead of subtracting its logarithm. The logarithm of the reciprocal of a number is called the *cologarithm* of the number.

For example, the cologarithm of 5, which is written colog 5, is the same as log $\frac{1}{5}$; that is,

$$\text{colog } 5 = \log \tfrac{1}{5} = \log 1 - \log 5 = 0 - \log 5,$$

or

$$\text{colog } 5 = -\log 5.$$

To avoid adding a negative mantissa in a group of positive mantissas, we may write $-\log 5$ as $10 - \log 5 - 10$; that is, we have

$$\text{colog } 5 = 10 - \log 5 - 10 = 10 - 0.6990 - 10 = 9.3010 - 10 = \bar{1}.3010.$$

Finding a Cologarithm. It is easy to write a cologarithm while looking at the corresponding logarithm in the table, for we need only subtract the logarithm from 10.0000 − 10.

For example, in finding colog 7, we have, as shown at the left,

$10.0000 - 10$	$9.999(10) - 10$
$\log 7 = \underline{0.8451}$	$\log 7 = \underline{0.8451}$
$\text{colog } 7 = 9.1549 - 10$	$\text{colog } 7 = 9.1549 - 10$
$= \bar{1}.1549$	$= \bar{1}.1549$

When we begin to subtract from the first figure at the right in 10.0000, the other figures all become 9's, as shown in the work at the right. In practice, therefore, we begin at the left and perform such a subtraction mentally by subtracting each figure except the right-hand one from 9.

In a simple division involving two numbers it is easier to subtract the logarithm of the divisor, but in a case like the one at the top of the page it is easier to add the cologarithm of the divisor than it is to add the three logarithms in the numerator and then subtract the logarithm of the divisor.

Exercises. Computing by Logarithms

1. Find the value of the expression given at the top of page 80, first by adding the three logarithms of the factors in the numerator and subtracting the logarithm of the divisor from the sum, and then by adding in one operation the three logarithms and the cologarithm of the divisor.

2. Find the value of $\frac{17.28 \times 6.25 \times 16.9}{1.44 \times 0.25 \times 1.3}$ first without using cologarithms and then by using them. Verify the result by canceling common factors and multiplying in the usual way.

3. Find the value of $\frac{3.14 \times 16.54^2}{144 \times 1.54^2}$ in the easiest way.

Find the value of each of the following expressions:

4. $\frac{8.64 \times 2.88}{0.144 \times 17.28}$.

5. $\frac{58.3 \times 0.98}{1.25 \times 1.6}$.

6. $\frac{25.6 \times 2.781}{14.87 \times 0.032}$.

7. $\sqrt{\frac{3.79 \times 38.6}{8.5 \times 109.8}}$.

8. $\sqrt{\frac{6 \times 4.3 \times 14.72}{0.8 \times 9.6 \times 19.7}}$.

9. $\sqrt[3]{\frac{78 \times 63 \times 227.5}{63.2 \times 32.05 \times 8.7}}$.

10. $\left(\frac{10.96 \times 0.99}{1.37 \times 79.2}\right)^{\frac{1}{2}}$.

11. $\left(\frac{13.58 \times 1.035}{0.73 \times 47.6}\right)^{\frac{1}{4}}$.

12. $\left(\frac{93.56 \times 59.86 \times 21.80}{147.3 \times 0.6 \times 32.75}\right)^{\frac{2}{3}}$.

13. $\left(\frac{9.6 \times 2.7 \times 0.073}{0.91 \times 3.46 \times 8.304}\right)^{\frac{3}{4}}$.

14. $\left(\frac{16.50 \times 1.02^2 \times 0.99}{2.75 \times 1.03^3 \times 0.78}\right)^{\frac{1}{2}}$.

15. $\left(\frac{2786 \times 1372 \times 4297}{2415 \times 8706 \times 9391}\right)^{\frac{2}{3}}$.

Find the value of each of the following expressions to the degree of accuracy permitted by the tables in this book:

16. 0.1407^{-3}.

17. $\frac{2}{3} \times 1.21^{-1.2}$.

18. $0.023^{-0.03}$.

19. $3.47^{-1.21}$.

20. $0.0273^{-1.453}$.

21. $0.243^{-1\frac{2}{3}}$.

22. The distance d in feet traveled by a freely-falling body in t seconds is given by the formula $d = \frac{1}{2} gt^2$. Taking the value of g as 32.16, find the distance that such a body will fall in 24.2 sec.

23. From the formula in Ex. 22 find the time that it will take a body to reach the ground from a height of 12,500 ft.

24. The time t in seconds for one complete vibration of a pendulum l centimeters long is given by the formula $t = 2\pi\sqrt{\frac{l}{g}}$. Taking the value of g as 980 and the value of π as 3.142, find the time of vibration for a pendulum that is 36.78 cm. in length.

25. The formula for the volume V of a cube in terms of the area of the total surface T is $V = (\frac{1}{6} T)^{\frac{3}{2}}$. From this formula find the volume of a cube which has a surface area of 24.95 sq. in.

26. From the formula $V = \frac{4}{3}\pi r^3$, which gives the volume V of a sphere of radius r, find the volume of a sphere that is 15.72 in. in diameter.

27. From the horse-power formula $H = \dfrac{d^2 n}{2.534}$ find the value of H when $d = 3.072$ and $n = 8$.

28. From the horse-power formula $T = \dfrac{33{,}000\, H}{2\pi r S}$ find the value of T when $H = 26.25$, $r = 6.3$, and $S = 1275$.

29. The formula $A = P(1 + r)^n$ gives the amount A of a principal of P dollars after n interest periods in which the interest is compounded at the rate r for each period. Using this formula, find the amount in 12 yr. of \$1275 invested at 5%, compounded annually.

30. From the formula in Ex. 29 find the rate r of interest, compounded annually, that must be received in order to have an investment double itself in 10 yr.; in 15 yr.; in 20 yr.; in 25 yr.

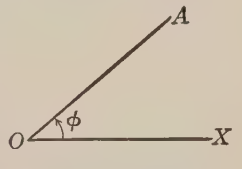
CHAPTER IV

TRIGONOMETRY

1. Trigonometric Functions

Nature of Trigonometry. The word *trigonometry* comes from three Greek words which mean "three," "angle," and "measure." It was formerly applied to the geometric methods used in measuring the triangle (a three-angled figure), but since the seventeenth century the name has come to be applied to the study of certain ratios relating to the angles of a triangle.

Generation of an Angle. Before considering these trigonometric ratios, however, it is necessary to understand what is meant in speaking of "the generation of an angle." When a line rotates about a point O in counterclockwise order from an initial position OX to a terminal position OA, as shown in this figure, we say that an angle is generated. We then speak of OX as the *initial side* of the $\angle XOA$ and of OA as the *terminal side.*

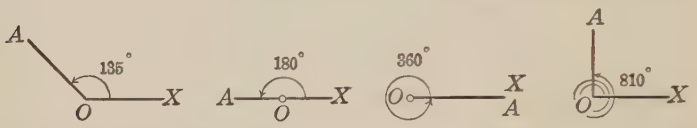


In designating angles the Greek letters ϕ (phi), θ (theta) and, ψ (psi) are most frequently used, but on occasion other letters, such as α (alpha), β (beta), γ (gamma), δ (delta), and so on, are used.

Positive and Negative Angles. When the angle is generated by a counterclockwise rotation, it is considered to be a positive angle. If the rotation is clockwise, the angle is considered negative.

Exceptions to this rule in surveying and navigation, and in the use of angles in gunnery, will be considered later.

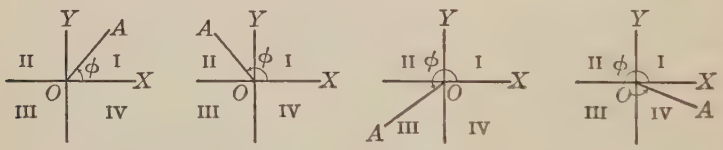
General Angles. It is evident that there need be no limit to the amount of rotation of the line which generates an angle. The following figures show how a line rotating counterclockwise from an initial position OX may generate positive angles of 135°, 180°, 360°, 810°, and so on, for angles of greater size:



A blade of an electric fan, for example, generates an angle of very large size in a few seconds when the fan is running at high speed.

In the applications of trigonometry the positive acute angle is the one most frequently used, the positive obtuse angle next, and the positive reflex angle (from 180° to 360°) next.

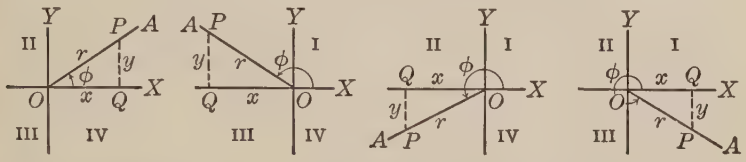
Angles and Quadrants. In the study of graphs on pages 36–38 we learned the convenience of using horizontal and vertical axes to divide the plane into four quadrants, and of defining the position of a point in the plane by means of its abscissa and ordinate. The same scheme is used with reference to angles; that is, as shown in these figures, when



the angle (ϕ in this case) is generated by the rotation of a line from the initial position OX, we say that ϕ is in the first quadrant if the terminal side OA lies in the first quadrant. Similarly, ϕ is said to be in the second, third, or fourth quadrants according to the position of OA with respect to these quadrants.

Thus, between 0° and 90° the angle is in the first quadrant, between 90° and 180° in the second, between 180° and 270° in the third, and between 270° and 360° in the fourth.

Conventional Symbolism. If we take any point P on the terminal side of the $\angle XOA$, or ϕ, the abscissa OQ is conventionally designated by x, the ordinate QP by y, and the distance OP (measured along the terminal side OA) by r.



The letter r is used to designate OP because it is the initial letter of "radius"; that is, ϕ is generated by the rotation of this radius about the origin O as center.

In considering the signs of x and y in trigonometry, we follow the same conventions that are used in connection with locating points for graphs; that is, in the four quadrants the signs of x and y vary as shown in the following table:

Quadrant	Sign of x	Sign of y
I	+	+
II	−	+
III	−	−
IV	+	−

Distance. The length of OP, or r, is called the *distance*; that is, it is the distance of the point P from the origin O. From the figures given above it is evident that in each of the four quadrants the triangle formed by the abscissa x, the ordinate y, and the radius r is a right triangle in which

$$r^2 = x^2 + y^2;$$

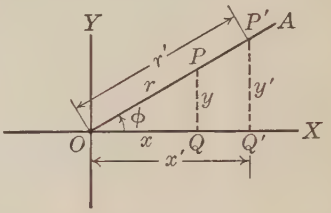
whence

$$r = \sqrt{x^2 + y^2}.$$

In this formula we see that x^2 and y^2 are always positive, whatever signs x and y may have. The sign of r is always taken as positive.

Besides locating a point like P in the above figures by means of the coordinates x and y, we can also locate it if we know the distance r and the size of ϕ. This method of locating points by means of distance and angle is extensively used in surveying.

Functions of an Angle. If on the terminal side of the $\angle XOA$, or ϕ, here shown we select any two points, $P(x, y)$ and $P'(x', y')$, and represent their distances from O by r and r' respectively, we see that $\triangle OQP$ is similar to $\triangle OQ'P'$, since ϕ is an angle which is common to both right triangles.



We therefore see that

$$\frac{y}{r} = \frac{y'}{r'}, \quad \frac{x}{r} = \frac{x'}{r'}, \quad \text{and} \quad \frac{y}{x} = \frac{y'}{x'},$$

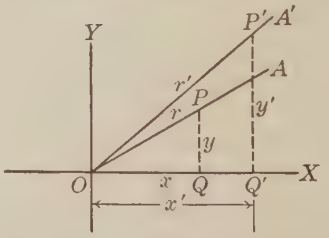
and that the reciprocals of these ratios $\left(\frac{r}{y}, \frac{r'}{y'}, \text{ and so on}\right)$ are also respectively equal. That is, a ratio like

$$\frac{y}{r}, \quad \text{or} \quad \frac{\text{ordinate}}{\text{distance}},$$

has the same value for P' that it has for P. Furthermore, since P and P' may be any points on the line OA, it is evident that the values of the ratios do not depend upon the lengths of x, y, and r.

Although ϕ in this case is an acute angle, it is easily seen that the discussion is also valid for angles in the other three quadrants.

If, however, we change the size of ϕ by letting OA rotate to the position OA', as shown in this figure, and then take P' on OA', we see that $\triangle OQP$ is no longer similar to $\triangle OQ'P'$, and that the ratio $\frac{y}{r}$ is no longer equal to $\frac{y'}{r'}$.



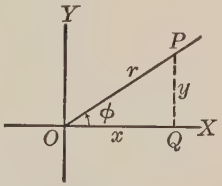
We thus see that the ratios $\frac{y}{r}, \frac{x}{r}, \frac{y}{x}$, and their reciprocals depend for their values upon the size of ϕ. For this reason such ratios are called *functions of the angle.*

The Six Important Functions. The six functions mentioned on page 86 have special names. For example, the ratio

$$\frac{y}{r}, \quad \text{or} \quad \frac{\text{ordinate}}{\text{distance}},$$

is called the *sine* of ϕ, which is usually written sin ϕ; that is,

$$\sin \phi = \frac{y}{r}.$$



Similarly, the other five important functions and the conventional abbreviations for them are as follows:

$$\textit{cosine} \text{ of } \phi = \frac{\text{abscissa}}{\text{distance}}, \quad \text{or} \quad \cos \phi = \frac{x}{r};$$

$$\textit{tangent} \text{ of } \phi = \frac{\text{ordinate}}{\text{abscissa}}, \quad \text{or} \quad \tan \phi = \frac{y}{x};$$

$$\textit{cotangent} \text{ of } \phi = \frac{\text{abscissa}}{\text{ordinate}}, \quad \text{or} \quad \cot \phi = \frac{x}{y};$$

$$\textit{secant} \text{ of } \phi = \frac{\text{distance}}{\text{abscissa}}, \quad \text{or} \quad \sec \phi = \frac{r}{x};$$

$$\textit{cosecant} \text{ of } \phi = \frac{\text{distance}}{\text{ordinate}}, \quad \text{or} \quad \csc \phi = \frac{r}{y}.$$

Of these six functions, the first three — the sine, the cosine, and the tangent — are the ones that are most frequently used in trigonometry.

Three other functions of ϕ, which are occasionally used in astronomy and navigation, are as follows:

$$\textit{versed sine} \text{ of } \phi = \frac{r - x}{r}, \quad \text{or} \quad \text{vers } \phi = \frac{r - x}{r};$$

$$\textit{coversed sine} \text{ of } \phi = \frac{r - y}{r}, \quad \text{or} \quad \text{covers } \phi = \frac{r - y}{r};$$

$$\textit{haversine} \text{ of } \phi = \frac{r - x}{2r}, \quad \text{or} \quad \text{haversin } \phi = \frac{r - x}{2r}.$$

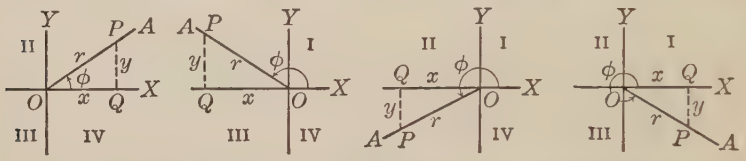
The last function is so named because it is half the versed sine of ϕ. We shall not need, however, to consider any of these three functions in our subsequent work.

Signs of the Functions. When we apply the definitions

$$\sin\phi = \frac{y}{r}, \qquad \tan\phi = \frac{y}{x}, \qquad \sec\phi = \frac{r}{x},$$

$$\cos\phi = \frac{x}{r}, \qquad \cot\phi = \frac{x}{y}, \qquad \csc\phi = \frac{r}{y},$$

to angles in any quadrant we see that the signs of the functions are not always the same. Applying the conventions



for the signs of x and y (r is always positive) in these figures, we see that the signs of the six functions change as follows:

Quadrant	$\sin\phi$	$\cos\phi$	$\tan\phi$	$\cot\phi$	$\sec\phi$	$\csc\phi$
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

Special Definitions. When ϕ is acute, as shown in the first of the above figures, x, y, and r are the sides of a right triangle containing ϕ, and hence

$$\sin\phi = \frac{\text{opposite side}}{\text{hypotenuse}}, \qquad \cot\phi = \frac{\text{adjacent side}}{\text{opposite side}},$$

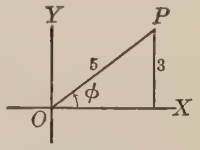
$$\cos\phi = \frac{\text{adjacent side}}{\text{hypotenuse}}, \qquad \sec\phi = \frac{\text{hypotenuse}}{\text{adjacent side}},$$

$$\tan\phi = \frac{\text{opposite side}}{\text{adjacent side}}, \qquad \csc\phi = \frac{\text{hypotenuse}}{\text{opposite side}}.$$

From these definitions $\cos\phi = \frac{OQ}{OP} = \sin OPQ$; that is, $\cos\phi$ is equal to the sine of the complement of ϕ. In fact, "cosine" is merely an abbreviated way of writing "the complement's sine," and similarly for cotangent and cosecant.

Exercises. Functions of an Angle

1. If ϕ is an angle in the first quadrant such that $\sin\phi = \frac{3}{5}$, what is the value of each of the other five functions?



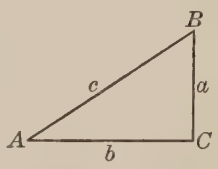
2. From the formula for r in terms of x and y, express $\sin\phi$ in terms of x and y; $\cos\phi$ in terms of r and y.

3. From the definition of $\cot\phi$ write a formula for x in terms of y and $\cot\phi$.

4. Write a formula for x in terms of r and $\sec\phi$; for r in terms of x and $\sec\phi$.

5. Write a formula for r in terms of y and $\csc\phi$; for y in terms of r and $\csc\phi$.

6. In this figure express $\sin A$ in terms of a and c; in terms of a and b; in terms of b and c.

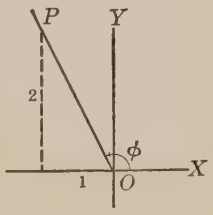


7. In the figure of Ex. 6 show that $\sin A = \cos B$, $\cos A = \sin B$, $\tan A = \cot B$, $\cot A = \tan B$, $\sec A = \csc B$, and $\csc A = \sec B$.

8. In the figure of Ex. 6 find the value of each of the six functions of $\angle A$ when $a = 6$ and $b = 8$; when $a = 5$ and $c = 13$; when $b = 3.9$ and $c = 8.9$.

9. In the figure of Ex. 6 find the value of each of the six functions of $\angle A$ when $b = \dfrac{2mn}{m-n}$ and $c = \dfrac{m^2+n^2}{m-n}$.

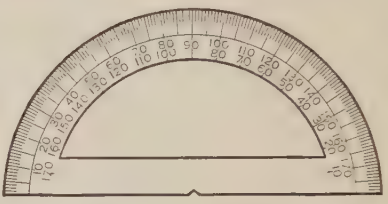
10. If ϕ is an angle in the second quadrant such that $\tan\phi = -2$, what is the value of each of the other five functions?



11. Of what functions of ϕ are $\cot\phi$, $\sec\phi$, and $\csc\phi$ the reciprocals?

12. Show that $\sin^2\phi + \cos^2\phi = 1$ is true when ϕ is in any quadrant.

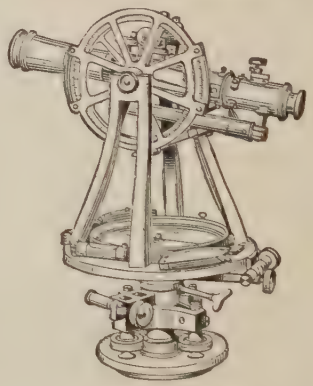
How Angles are Measured. Ordinarily, in measuring the sizes of angles in plans, maps, and similar cases we use a *protractor*. The accuracy with which the angle can be measured depends upon the size and the markings of the instrument, but for ordinary work it is possible to measure to the nearest $\frac{1}{4}°$, or 15′, with a good protractor.



Protractors are made in various sizes and shapes, and inexpensive ones can be purchased at most stationery stores. With small protractors it is not possible to measure so accurately; in fact, with a protractor of the type here shown it would not be possible to measure more accurately than to the nearest 30′.

The protractor is also used in constructing angles of any required size. While the results are, of course, approximate, they are often accurate enough for practical purposes.

Surveyors and others who have occasion to measure angles out of doors generally use a *transit* like the one shown below. The scale on the circle at the left of the telescope measures the angle through which the telescope is moved up or down in sighting at an object. The scale on the circle at the base of the supports of the telescope shows the angle through which the telescope is moved to the right or left.

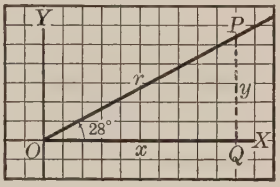


By means of vernier attachments and magnifying glasses it is possible to measure angles to a high degree of accuracy.

The degree of accuracy required in measuring an angle depends, of course, upon the nature of the work. In astronomical work, where great accuracy is necessary, transits are fitted with much larger scales.

Finding the Functions of an Angle. Although the inaccuracy of instruments and observations makes it impossible to secure absolute precision in any kind of measurement, we can, if we wish, find values for the different functions of an angle to a fair degree of accuracy by measuring a figure.

For example, suppose that we wish to find the functions of an angle of 28°.



With the protractor we may first construct an angle of 28° on squared paper, as here shown, and then on the terminal side of the angle we may choose a point $P(x, y)$ such that x or y is a convenient integer. In this case $x = 10$ is convenient, and we see that $y = 5.3$, approximately. Then, by measuring OP, or r, with a ruler, or from the relation

$$r = \sqrt{x^2 + y^2},$$

we find that $r = 11.3$. In finding the functions of the angle to the nearest 0.01 in each case we then have

$$\sin 28° = \frac{y}{r} = \frac{5.3}{11.3} = 0.47 \qquad \cot 28° = \frac{x}{y} = \frac{10}{5.3} = 1.89$$

$$\cos 28° = \frac{x}{r} = \frac{10}{11.3} = 0.88 \qquad \sec 28° = \frac{r}{x} = \frac{11.3}{10} = 1.13$$

$$\tan 28° = \frac{y}{x} = \frac{5.3}{10} = 0.53 \qquad \csc 28° = \frac{r}{y} = \frac{11.3}{5.3} = 2.13$$

It is evident that the degree of accuracy of the results depends upon the degree of accuracy of the measurements. Since with the ordinary protractor, which is graduated only to degrees, we could not construct satisfactorily an angle of 28° 37′, the method is limited. Furthermore, in the case of angles near 0° or 90° a slight error in the measurements of x, y, and r, or in the construction of the angle, would produce a large error in the results for the tangent, cotangent, secant, and cosecant.

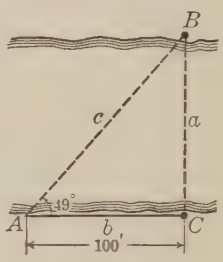
For example, $\tan 89° 42′ = 191.0$ and $\tan 89° 48′ = 286.5$; that is, for angles of this size the tangent is changing so rapidly that the above method of finding the functions would not yield reliable results.

Exercises. Finding Functions

By constructing the angle and measuring its sides, as on page 91, find the six functions of an angle of

1. 40°.	3. 70°.	5. 65°.	7. 62°.	9. 43°.
2. 50°.	4. 35°.	6. 47°.	8. 71°.	10. 39°.

11. In finding the width of a river a stake C was set at a point directly opposite a tree B on the farther bank. A transit was then set up at a point A that was 100 ft. from C on a line at right angles to BC, and $\angle A$ was then measured and found to be 49°. By constructing $\triangle ACB$ on squared paper to a convenient scale, find the approximate width of the river.



12. Taking tan 49° as 1.1504, show how to find the width of the river in Ex. 11 without using a drawing, and find the width by this method.

13. In the triangle constructed in Ex. 11 find the other five functions of $\angle A$.

14. For safety and ease of climbing, a ladder raised against a wall of a building should make an angle with the horizontal of about 75°. By constructing the angle find cos 75° and sin 75°.

15. From Ex. 14 find how far the foot of a 30-foot ladder should be placed from the wall of a building in order to make an angle of 75° with the horizontal.

16. In Ex. 15 how high does the ladder reach on the wall of the building?

17. To meet the requirements mentioned in Ex. 14, how long a ladder would be needed to reach a window sill 50 ft. above the ground?

Angles Determined by Functions. We have seen how by constructing an angle of a given size and measuring certain lines we can find the functions of this angle at least approximately. A second problem is to construct the angle when we know one of its functions; for example, given that $\sin \phi = 0.6$, construct the angle ϕ.

In actual practice both of the above problems are solved by the use of tables. That is, in the first case, if we need to know the value of tan 75°, we look it up in a table such as will be described later. Similarly, if we find that tan $\phi = 3.7321$ and wish to know the number of degrees in ϕ, we again consult the table. To understand the meanings of the functions and their relations to the size of an angle, however, it is desirable to know how to solve both of these problems by methods that do not depend upon tables.

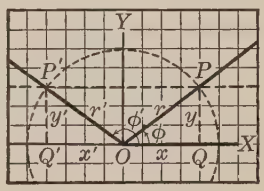
In constructing the angle which has 0.6, or $\frac{6}{10}$, as its sine we have

$$\sin \phi = \frac{y}{r};$$

whence

$$\frac{y}{r} = \frac{6}{10} = \frac{3}{5},$$

or $$y = 3 \text{ and } r = 5.$$



Using squared paper, as shown in the figure above, we first draw a circle with the origin O as center and a radius of 5 units. We then draw a line parallel to the x axis and 3 units above it, and find that this line cuts the circle at the two points P and P'. Drawing lines from O through P and P', we then have two angles, ϕ and ϕ', each of which has the sine 0.6, as required.

Furthermore, ϕ lies in the first quadrant and ϕ' in the second, which is what we should expect since we know from page 88 that the sine in both of these quadrants is positive.

We might have constructed an indefinite number of angles having the sine 0.6, because $360° + \phi$, $720° + \phi$, and so on, have the same terminal side as ϕ, and the same is true for ϕ'. In general, however, in such a case we need consider only ϕ and ϕ'.

Exercises. Angles and Functions

Using the method of page 93, construct ϕ when

1. $\sin\phi = \frac{2}{3}$.
2. $\cos\phi = \frac{2}{3}$.
3. $\tan\phi = \frac{2}{3}$.
4. $\cot\phi = \frac{2}{3}$.
5. $\sec\phi = \frac{3}{2}$.
6. $\sin\phi = -\frac{3}{5}$.
7. $\cos\phi = -\frac{3}{5}$.
8. $\tan\phi = -\frac{1}{3}$.
9. $\cot\phi = -\frac{3}{5}$.
10. $\sec\phi = -2$.
11. $\sin\phi = 1$.
12. $\sin\phi = -1$.
13. $\tan\phi = -2$.
14. $\cot\phi = -2$.
15. $\sec\phi = -3$.

In Ex. 2, $x = 2$ and $r = 3$, and hence P and P' may be located by finding the intersection of a line and a circle. In Ex. 3, $y = \pm 2$ and $x = \pm 3$, and hence P and P' may be located by these coordinates.

16. Construct ϕ when $\sin\phi = \frac{3}{4}$. Find the other five functions of ϕ.

Since $r^2 = x^2 + y^2$, we have $x = \sqrt{r^2 - y^2} = \sqrt{16 - 9} = \sqrt{7} = \pm 2.65$, approximately. The functions are then found as on page 91.

Construct ϕ and find the other five functions when

17. $\sin\phi = \frac{1}{3}$.
18. $\cos\phi = \frac{1}{4}$.
19. $\tan\phi = \frac{2}{5}$.
20. $\cot\phi = \frac{7}{8}$.
21. $\csc\phi = 4$.
22. $\sin\phi = -\frac{3}{8}$.
23. $\sec\phi = -2\frac{1}{2}$.
24. $\tan\phi = -\frac{1}{2}$.
25. $\sec\phi = -3\frac{1}{2}$.

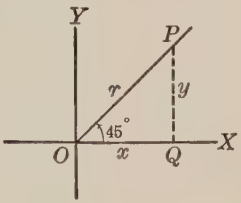
26. When $\tan\phi = \sqrt{2}$, show that $\cot\phi$ is half as large as $\tan\phi$, and find the other four functions of ϕ.

27. By dividing the length of a vertical pole by the length of its horizontal shadow, the tangent of the angle of elevation of the sun is found to be 1.24. What is the height of a tower which casts a shadow 62.3 ft. long at the same time?

The "angle of elevation of the sun" means the angle that the sun's rays, which are regarded as parallel, make with the horizontal.

28. When a fence post that is 51 in. high casts a horizontal shadow 37 in. long, what is the tangent of the angle of elevation of the sun? What is the height of a tree which casts a shadow 22.6 ft. long at the same time?

Functions of 45°. In finding the functions of 45° from the accompanying figure we need simply observe that $\triangle OQP$ is an isosceles right triangle, and hence



$$\angle O = \angle P = 45°,$$

and $$x = y.$$

Then $$r = \sqrt{x^2 + y^2},$$

and so $$r = \sqrt{2\,x^2} = x\sqrt{2},$$

or $$r = \sqrt{2\,y^2} = y\sqrt{2}.$$

Then $$\sin 45° = \frac{y}{r} = \frac{y}{y\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = 0.707,$$

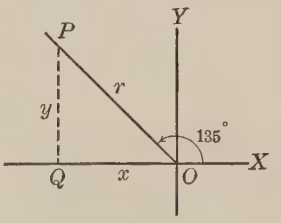
and $$\cos 45° = \frac{x}{r} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} = 0.707.$$

In the same way, for the other functions we have

$$\tan 45° = \frac{y}{x} = 1 \qquad \sec 45° = \frac{r}{x} = \frac{x\sqrt{2}}{x} = \sqrt{2} = 1.414$$

$$\cot 45° = \frac{x}{y} = 1 \qquad \csc 45° = \frac{r}{y} = \frac{y\sqrt{2}}{y} = \sqrt{2} = 1.414$$

Functions of 135°. Similarly in this figure, in finding the functions of 135°, we have



$$\angle POQ = 180° - 135° = 45°,$$

and so $$x = y,$$

and $$r = x\sqrt{2} = y\sqrt{2},$$

as in the case of an angle of 45°, above. We therefore have

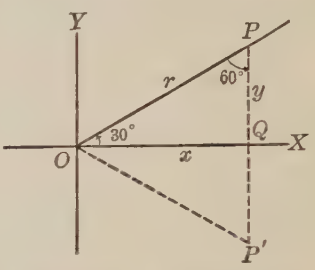
$$\sin 135° = \frac{y}{r} = 0.707 \qquad \cot 135° = -\frac{x}{y} = -1$$

$$\cos 135° = -\frac{x}{r} = -0.707 \qquad \sec 135° = -\frac{r}{x} = -1.414$$

$$\tan 135° = -\frac{y}{x} = -1 \qquad \csc 135° = \frac{r}{y} = 1.414$$

That is, except for certain signs, the functions of 135° are the same as those of 45°.

Functions of 30° and 60°. We can also find the functions of 30° and 60° without resorting to measurement. In this figure we see that the $\triangle OP'P$ is equilateral, and hence OQ is the perpendicular bisector of PP'.



Then $y = \frac{1}{2}r$,

and
$$x = \sqrt{r^2 - y^2}$$
$$= \sqrt{r^2 - \tfrac{1}{4}r^2}$$
$$= \tfrac{1}{2}r\sqrt{3}.$$

For the functions of 30° and the functions of its complement, 60°, we therefore have

$$\sin 30° = \cos 60° = \frac{y}{r} = \frac{\frac{1}{2}r}{r} = \frac{1}{2} = 0.5$$

$$\cos 30° = \sin 60° = \frac{x}{r} = \frac{\frac{1}{2}r\sqrt{3}}{r} = \frac{\sqrt{3}}{2} = 0.866$$

$$\tan 30° = \cot 60° = \frac{y}{x} = \frac{\frac{1}{2}r}{\frac{1}{2}r\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = 0.577$$

$$\cot 30° = \tan 60° = \frac{x}{y} = \frac{\frac{1}{2}r\sqrt{3}}{\frac{1}{2}r} = \sqrt{3} = 1.732$$

$$\sec 30° = \csc 60° = \frac{r}{x} = \frac{r}{\frac{1}{2}r\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = 1.155$$

$$\csc 30° = \sec 60° = \frac{r}{y} = \frac{r}{\frac{1}{2}r} = 2$$

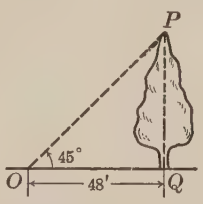
By the method used on page 95 we can easily find the functions of 180° − 30°, or 150°, and of 180° − 60°, or 120°. The signs of the cosines, tangents, cotangents, and secants of these angles are negative.

The following arrangement is helpful in learning the sines and cosines of 30°, 45°, and 60°, functions that we shall frequently need to use in subsequent work:

$$\sin 30° = \tfrac{1}{2}\sqrt{1} \qquad \sin 45° = \tfrac{1}{2}\sqrt{2} \qquad \sin 60° = \tfrac{1}{2}\sqrt{3}$$
$$\cos 30° = \tfrac{1}{2}\sqrt{3} \qquad \cos 45° = \tfrac{1}{2}\sqrt{2} \qquad \cos 60° = \tfrac{1}{2}\sqrt{1}$$

Exercises. Functions of 30°, 45°, and 60°

1. From a point O on the ground the angle of elevation ($\angle QOP$ in the figure) of the top of a tree 48 ft. away is found to be 45°. By using one of the properties learned in geometry about an isosceles triangle, find the height of the tree. Verify the result by using the tangent of $\angle O$.



2. In Ex. 1 what would have been the height of the tree if $\angle O$ had been 30°? if $\angle O$ had been 60°?

3. In Ex. 2 find the distance from the observer at O to the top P of the tree in each case.

4. Taking $\sqrt{2}$ as 1.4142, find the functions of 45°.

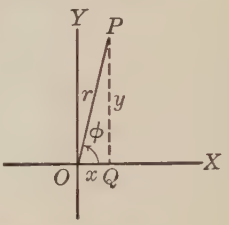
5. Taking $\sqrt{3}$ as 1.7321, find the functions of 30° and the functions of 60°.

6. In one size of a draftsman's triangle, which has acute angles of 30° and 60°, the side opposite the angle of 30° is 8 in. long. Find the length of the hypotenuse and the length of the other side of the triangle.

In any of the four quadrants in which it is permissible, construct ϕ when

7. $\sin \phi = 0.7071$.
8. $\tan \phi = 1.0000$.
9. $\sec \phi = 1.4142$.
10. $\cos \phi = 0.7071$.
11. $\cot \phi = 1.0000$.
12. $\csc \phi = 1.4142$.
13. $\cos \phi = -0.7071$.
14. $\cot \phi = -1.0000$.
15. $\cos \phi = 0.5000$.
16. $\cos \phi = 0.8660$.
17. $\cot \phi = 0.5774$.
18. $\tan \phi = 1.7321$.
19. $\sec \phi = 1.1547$.
20. $\sec \phi = 2.0000$.
21. $\cos \phi = -0.8661$.
22. $\sec \phi = -1.1547$.

Functions of 90°. As stated on page 91, finding the functions by measurement is not reliable when ϕ approaches either 0° or 90°. From this figure it is evident that as ϕ approaches 90°, x becomes smaller and smaller and the length of y approaches nearer and nearer the length of r, until when $\phi = 90°$ we have



$$x = 0 \quad \text{and} \quad r = y.$$

From the definitions of the functions, we then have

$$\sin 90° = \frac{y}{r} = \frac{y}{y} = 1 \qquad \cot 90° = \frac{x}{y} = \frac{0}{y} = 0$$

$$\cos 90° = \frac{x}{r} = \frac{0}{r} = 0 \qquad \csc 90° = \frac{r}{y} = \frac{r}{r} = 1$$

We also have $\tan 90° = \frac{y}{x} = \frac{y}{0}$ and $\sec 90° = \frac{r}{x} = \frac{r}{0}$. Strictly speaking, such forms are meaningless because division by zero is not allowed. However, if y does not decrease and x becomes smaller and smaller, approaching zero as a limit, the value of $\frac{y}{x}$ becomes larger and larger. We therefore say that "as x approaches zero as its limit, $\frac{y}{x}$ increases without limit," and express this in symbols by writing "as $x \to 0$, $\frac{y}{x} \to \infty$." Hence, as $\phi \to 90°$, we have

$$\tan \phi \to \infty \quad \text{and} \quad \sec \phi \to \infty.$$

This is sometimes abbreviated to "$\tan 90° = \infty$ and $\sec 90° = \infty$."

Functions of 0°. If ϕ in the above figure decreases, it is evident that when $\phi = 0$ we have $y = 0$ and $r = x$. Hence

$$\sin 0° = \frac{y}{r} = \frac{0}{r} = 0 \qquad \cot 0° = \frac{x}{y} = \frac{x}{0}, \text{ or } \cot \phi \to \infty$$

$$\cos 0° = \frac{x}{r} = \frac{x}{x} = 1 \qquad \sec 0° = \frac{r}{x} = \frac{x}{x} = 1$$

$$\tan 0° = \frac{y}{x} = \frac{0}{x} = 0 \qquad \csc 0° = \frac{r}{y} = \frac{r}{0}, \text{ or } \csc \phi \to \infty$$

Exercises. Functions of Special Angles

1. Since $\sin \phi = \frac{y}{r}$, we see that $\frac{1}{\sin \phi} = \frac{r}{y} = \csc \phi$. Using this relation, find csc 0° from sin 0°. In the same way, from tan 0° and cos 0° find cot 0° and sec 0° respectively.

2. As in Ex. 1, find cot 90°, sec 90°, and csc 90° from their reciprocals.

Draw suitable figures and find the following functions:

3. sin 180°. 6. cot 180°. 9. sin 270°. 12. cot 270°.

4. cos 180°. 7. sec 180°. 10. cos 270°. 13. sec 270°.

5. tan 180°. 8. csc 180°. 11. tan 270°. 14. csc 270°.

15. Draw a suitable figure and from it find the six functions of 150°.

16. As in Ex. 15, find the functions of 120°.

17. Draw a suitable figure on squared paper, instead of using the figure on page 95, and find the functions of 135°.

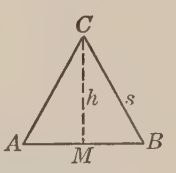
Draw suitable figures and find the following functions:

18. sin 225°. 21. sin 360°. 24. sin 210°. 27. sin 240°.

19. cos 225°. 22. cos 360°. 25. cos 210°. 28. cos 240°.

20. tan 225°. 23. tan 360°. 26. tan 210°. 29. tan 240°.

30. In this equilateral triangle in which CM (or h) is perpendicular to AB, AM is what part of the side s? Express sin A in terms of h and s. Express h in terms of s and sin A.



31. In the figure of Ex. 30, what is the value of sin ACM? of tan ACM?

32. Find the height of an equilateral triangle that is 10 in. on a side and find the length of the side of an equilateral triangle that is 10 in. high.

Tables of Functions. In practical work we do not stop to find a trigonometric function by the methods which we have just studied, because such methods are too inaccurate and time-wasting. Instead we use tables in which the functions have been computed to a certain number of decimal places, as in the four-place tables given on pages 362–373. These tables are called *tables of natural functions*, in order to distinguish them from the tables of logarithmic functions which we shall consider later.

In using the table on pages 362 and 363 to find sin 58°, for example, we look in the left-hand column on page 363 for **58**, and then at the right, in the column headed **0′** (or **0.0°**), we find .8480; that is, sin 58° = 0.8480 to the nearest 0.0001. The tables of cosines, tangents, cotangents, secants, and cosecants which are given on pages 364–373 are used in a similar manner.

For the present we shall need to find functions of angles that are given to the nearest 1° only. The further use of the tables will be explained later when the need arises.

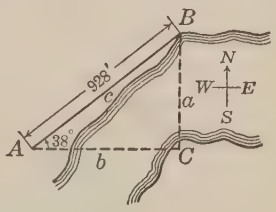
Exercises. Tables of Natural Functions

From the tables on pages 362–373 find

1. sin 8°.	**10.** sin 74°.	**19.** cot 11°.	**28.** sin 0°.
2. sin 13°.	**11.** cos 14°.	**20.** tan 69°.	**29.** cos 0°.
3. sin 23°.	**12.** sin 76°.	**21.** tan 21°.	**30.** sin 1°.
4. sin 40°.	**13.** cos 24°.	**22.** cot 39°.	**31.** tan 2°.
5. cos 85°.	**14.** sin 66°.	**23.** tan 45°.	**32.** csc 65°.
6. cos 76°.	**15.** cos 30°.	**24.** cot 45°.	**33.** csc 12°.
7. cos 68°.	**16.** sin 45°.	**25.** sec 60°.	**34.** sec 78°.
8. cos 60°.	**17.** cot 66°.	**26.** csc 30°.	**35.** csc 34°.
9. cos 79°.	**18.** tan 75°.	**27.** sec 45°.	**36.** sec 48°.

Practical Uses of the Functions. The following examples show how the functions of angles from 0° to 90° are used:

1. In measuring the distance BC across a river, as here shown, a surveyor set up his transit at A, due west of C. By measuring he then found that $\angle A = 38°$ and that $AB = 928$ ft. How wide was the river at BC?



Since $\sin A = \frac{a}{c}$,

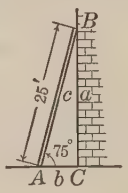
we have $a = c \sin A$.

From page 362, $\sin 38° = 0.6157$, and so

$a = c \sin A = 928 \times 0.6157 = 571.3696$.

Since we used a four-place table, only the first four figures of this result can be correct. Furthermore, in such a case we should probably measure only to the nearest 0.1 ft., and hence we say that the width is 571.4 ft.

2. In placing a 25-foot ladder against a wall so that it shall make an angle of 75° with the ground, how far from the wall should the foot of the ladder be placed?



Since $\cos A = \frac{b}{c}$,

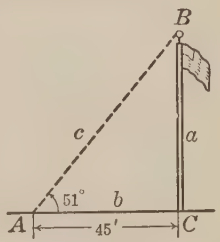
we have $b = c \cos A$.

From page 365, $\cos 75° = 0.2588$, and so

$b = 25 \times 0.2588 = 6.4700$.

Hence to the nearest 0.1 ft. the ladder should be 6.5 ft. from the wall.

3. If from a point on the ground 45 ft. from the foot of a flagpole the angle of elevation of the top is 51°, what is the height of the pole?



Since $\tan A = \frac{a}{b}$,

we have $a = b \tan A$.

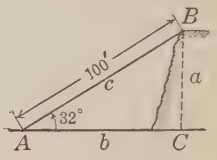
From page 367, $\tan 51° = 1.2349$, and so

$a = 45 \times 1.2349 = 55.5705$.

Hence to the nearest 0.1 ft. the height of the pole is 55.6 ft.; that is, we may reasonably assume that the length of b was measured to this degree of accuracy.

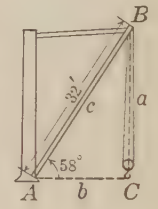
Exercises. Practical Uses of the Functions

1. In finding the height of a cliff it was not possible to drop a plumb line from the top, and so a 100-foot tape was stretched to a point A, as shown in the figure. If $\angle A$ was found to be 32°, what was the height of the cliff to the nearest foot?



2. With a transit placed 5 ft. above the level of the water on a bank of a river 2650 ft. wide, a surveyor finds that the angle of elevation of the top of a cliff which rises vertically on the opposite bank is 5°. What is the height of the cliff?

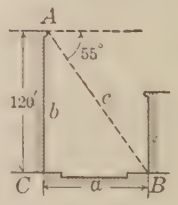
3. When the 32-foot arm of a derrick makes an angle of 58° with the horizontal, as shown in this figure, how far is the pulley C from the base of the derrick?



4. Solve Ex. 3, assuming that the arm is raised until $\angle A = 87°$.

5. A submarine cable which runs 38° north of east from a shore station is known to be broken at a point 195 mi. from the station. How far to the east and how far to the north of the station is the break in the cable?

6. From the top of a building 120 ft. high the angle of depression of the opposite side of the street is observed to be 55°. How wide is the street, including the sidewalks?

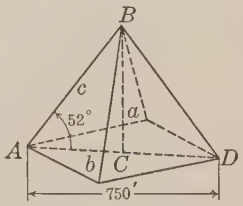


Since the angle of depression at A is 55°, we have $\angle CAB = 90° - 55° = 35°$. By using tan 35°, how can we now find the value of a? We might also observe that $\angle ABC = 55°$, and then solve for a.

7. A ship that is sailing on a course running 49° north of east is 32 mi. further north at the end of 3 hr. What is the speed of the ship in miles per hour?

8. With a transit set on the roof of a building so that the instrument is 58 ft. 6 in. above the ground, the angle of depression to the base of a telegraph pole near the building is observed to be 43°. How far is the pole from the building?

9. This figure represents the Great Pyramid of Egypt. By measuring the square base it is found that the diagonal AD is 750 ft. long and that the angle of elevation of B at A is 52°. Find the height of the pyramid.



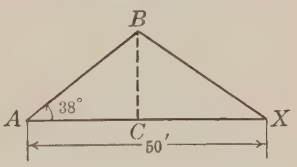
10. At a height of 6200 ft. an aviator estimates that the angle of depression of a building is 38°. How far is the building from a point on the ground directly beneath the plane? How far in a straight line is the building from the aviator?

11. After 1600 ft. of cable has been unreeled in sending up a captive balloon, how high is the balloon when the cable makes an angle of 82° with the ground?

The cable will, of course, sag considerably; if we assume it to be straight, however, the result will be sufficiently accurate for our purposes.

12. In a rectangle which is 11.8 in. long the diagonal makes an angle of 7° with the base. Find the length of the diagonal and the area of the rectangle.

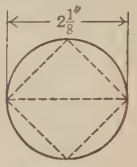
13. In planning a roof which is to have a span (AX in the figure) of 50 ft. and an angle of slope of 38°, what length (AB) should be allowed for the rafters? How high is the peak of the roof above the eaves?



14. A straight road runs at an average gradient of 3° for a distance, measured along the road, of $2\frac{1}{2}$ mi. How many feet higher is one end of the road than the other?

The *gradient* is the angle of slope with the horizontal.

15. Find, to the nearest 0.001 in., the length of the side of the greatest square that can be milled on the round shaft here shown; on a shaft $3\frac{1}{16}$ in. in diameter; on a shaft 8.270 in. in diameter.



From the figure it is obvious that the greatest square is an inscribed square.

16. A regular hexagon is inscribed in a circle with a diameter of 10.8 in. Find the distance from the center to a side of the hexagon.

How many degrees are there in each central angle? in half this angle? How may we now find x?

17. In Ex. 16 find the area of the hexagon.

How do the radius and the side of a regular inscribed hexagon compare in length?

18. Eight evenly-spaced holes, each 1 in. inside the rim, are to be drilled in a cylinder head which is 28 in. in diameter. Find the distance between the centers of the holes.

On line **22** in the column headed **0.5°** on page 362 we find that $\sin 22.5° = 0.3827$.

19. Solve Ex. 18, assuming that twelve holes are to be drilled in the cylinder head.

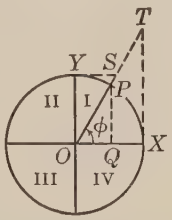
20. Find the perimeter of a regular decagon inscribed in a circle with a diameter of 20 in.

Similarly, half of each side of a polygon of 90 sides inscribed in a circle of diameter 1 is $\frac{1}{2}\sin 2°$, and hence the perimeter is $90 \times 2 \times (\frac{1}{2} \times 0.0349)$, or 3.141. Since the diameter is 1, this result is an approximate value for π in the formula $C = \pi d$.

21. In Ex. 20 show that a formula for the approximate area A of a regular decagon inscribed in a circle of radius r is $A = 2.9389\, r^2$. Find the area of the decagon and the area of that part of the circle not covered by the decagon.

Line Values of the Functions. In the trigonometric ratios

$$\frac{y}{r}, \quad \frac{x}{r}, \quad \frac{y}{x}, \quad \frac{x}{y}, \quad \frac{r}{x}, \quad \text{and} \quad \frac{r}{y},$$

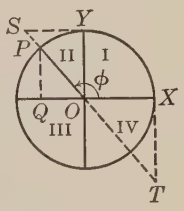


it is evident that if we can make each denominator equal to 1, we may represent the functions by the lines y, x, and r. For this purpose we use a circle with the radius 1, which is called a *unit circle*. Thus, in the figure above, in which ϕ is an angle in the first quadrant, we see that

$$\sin\phi = QP \qquad \tan\phi = XT \qquad \sec\phi = OT$$
$$\cos\phi = OQ \qquad \cot\phi = YS \qquad \csc\phi = OS$$

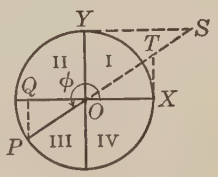
That is, $\sin\phi = \frac{QP}{OP} = \frac{QP}{1} = QP$, $\tan\phi = \frac{XT}{OX} = \frac{XT}{1} = XT$, and so on.

We thus see, as on page 88, that when ϕ is in the first quadrant all the functions are positive.



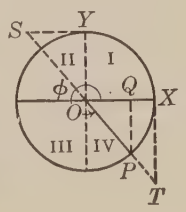
When ϕ is in the second quadrant we see that, as above, $\sin\phi = QP$, $\cos\phi = OQ$, and so on. In this case we see that only the sine and cosecant are positive.

Since OT is measured in the opposite direction from OP, OT is negative.



When ϕ is in the third quadrant, as shown in this figure, we see that only the tangent and cotangent are positive.

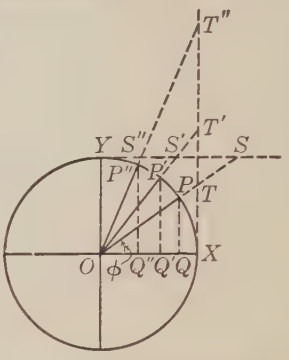
Since XT is a tangent to the circle and OS is a secant, we see the reason for these names.



When ϕ is in the fourth quadrant, as shown in this figure, we see that only the cosine and secant are positive.

We also see in the first figure that the sine of the complement of ϕ (that is, $\sin OPQ$) is OQ, which is the line that represents $\cos\phi$, and similarly for the cotangent and the cosecant of this angle.

Changes in the Functions of an Acute Angle. If we let $\angle XOP$, or ϕ, in this figure increase gradually to 90°, we see that sin ϕ, or QP, increases to $Q'P'$, $Q''P''$, and so on, until it coincides with OY, which is a radius of the unit circle.



That is, sin ϕ increases from 0 when $\phi = 0°$ to 1 when $\phi = 90°$, and hence the *limiting values* of sin ϕ are 0 and 1.

Similarly, XT and OT gradually increase in length, while OQ, YS, and OS gradually decrease. That is,

As ϕ increases from 0° to 90°, sin ϕ, tan ϕ, and sec ϕ also increase, while cos ϕ, cot ϕ, and csc ϕ decrease.

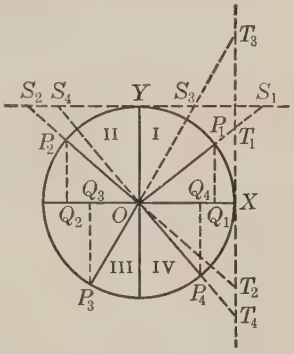
From the figure and from what we have already learned on page 98 about the functions of 0° and 90°, we may make the following table to show the nature of the changes in the functions of ϕ as the angle increases in size from 0° to 90°:

Function	Nature of Change	From	To
sin ϕ	increases	0	1
cos ϕ	decreases	1	0
tan ϕ	increases	0	∞
cot ϕ	decreases	∞	0
sec ϕ	increases	1	∞
csc ϕ	decreases	∞	1

That is, when we say that tan ϕ increases to ∞, we mean that tan ϕ increases without limit. We see from the figure that as $\phi \to 90°$, XT becomes longer and longer until, when $\phi = 90°$, XT is parallel to OT, which coincides with OY, and hence XT and OT cannot meet.

From the figure we also see that sin ϕ and cos ϕ can never exceed 1 (the radius of the circle), that sec ϕ and csc ϕ are never less than 1, and that tan ϕ and cot ϕ may have any values from 0 to ∞.

Changes in sin ϕ. A study of the line values of the functions shows how they change as the angle increases from 0° to 360°. For example, from this figure we see that when ϕ is in the first quadrant, sin ϕ, or Q_1P_1, is positive and increases from 0 to 1. In the second quadrant sin ϕ, or Q_2P_2, is still positive, and decreases from 1 to 0. In the third quadrant sin ϕ, or Q_3P_3, is negative and its numerical value increases from 0 to 1. In the fourth quadrant sin ϕ, or Q_4P_4, is negative and decreases in numerical value from 1 to 0. That is, the numerical value of sin ϕ varies from 0 to 1, while the total range is from $+1$ to -1.

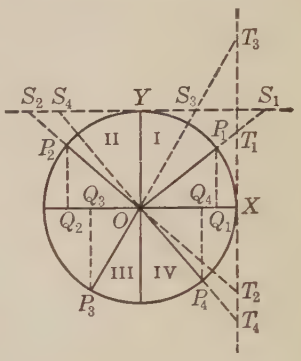


In the third quadrant sin ϕ *decreases* from 0 to -1, but its numerical value *increases* from 0 to 1.

Changes in cos ϕ. When ϕ is in the first quadrant, cos ϕ, or OQ_1, is positive and decreases from 1 to 0. In the second quadrant cos ϕ, or OQ_2, becomes negative and increases in numerical value from 0 to 1. In the third quadrant cos ϕ, or OQ_3, remains negative and decreases in numerical value from 1 to 0, while in the fourth quadrant cos ϕ, or OQ_4, becomes positive and increases from 0 to 1. The numerical value of cos ϕ therefore varies from 1 to 0, the total range being from $+1$ to -1.

Changes in tan ϕ. Similarly, in the first quadrant tan ϕ, or XT_1, is positive and increases from 0 to ∞ ; in the second tan ϕ, or XT_2, becomes negative and its numerical value decreases from ∞ to 0; in the third tan ϕ, or XT_3, is positive and increases from 0 to ∞ ; and in the fourth tan ϕ, or XT_4, is negative and its numerical value decreases from ∞ to 0.

Changes in cot ϕ. As on page 107, we see from the figure that in the first quadrant cot ϕ, or YS_1, is positive and decreases from ∞ to 0; in the second quadrant cot ϕ, or YS_2, is negative and the numerical value increases from 0 to ∞; and in the third and fourth quadrants cot ϕ, or YS_3 and YS_4, has the same signs and the same changes that it does in the first two quadrants.



The changes in cot ϕ, sec ϕ, and csc ϕ are relatively less important than those considered on page 107.

Changes in sec ϕ and csc ϕ. From the figure we see that the changes in the numerical values of sec ϕ and csc ϕ are opposite to each other. That is, in the first quadrant sec ϕ, or OT_1, is positive and increases from 1 to ∞, while csc ϕ, or OS_1, is positive and decreases from ∞ to 1. In the second quadrant sec ϕ, or OT_2, is negative and its numerical value decreases from ∞ to 1, while csc ϕ, or OS_2, is positive and increases from 1 to ∞.

Summary. We may now construct the following table:

Function	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
sin ϕ	0 to +1	+1 to 0	0 to −1	−1 to 0
cos ϕ	+1 to 0	0 to −1	−1 to 0	0 to +1
tan ϕ	0 to $+\infty$	$-\infty$ to 0	0 to $+\infty$	$-\infty$ to 0
cot ϕ	$+\infty$ to 0	0 to $-\infty$	$+\infty$ to 0	0 to $-\infty$
sec ϕ	+1 to $+\infty$	$-\infty$ to −1	−1 to $-\infty$	$+\infty$ to +1
csc ϕ	$+\infty$ to +1	+1 to $+\infty$	$-\infty$ to −1	−1 to $-\infty$

It is evident that the functions always change sign in passing through 0 or ∞, and in the table the signs before ∞ simply indicate the direction from which the value is reached. That is, in the first quadrant tan ϕ is positive and increases from 0 to ∞, while in the second it is negative and is changing from a very large negative value to 0.

Exercises. Changes in Functions

Using a unit circle, represent the following functions by lines:

1. sin 105°.	7. sin 220°.	13. sin 310°.	19. sin 260°.
2. cos 130°.	8. cos 245°.	14. cos 305°.	20. cos 180°.
3. tan 160°.	9. tan 230°.	15. tan 360°.	21. tan 180°.
4. cot 135°.	10. cot 210°.	16. cot 360°.	22. cot 270°.
5. sec 120°.	11. sec 225°.	17. sec 315°.	23. sec 180°.
6. csc 170°.	12. csc 260°.	18. csc 320°.	24. csc 270°.

25. Make up a table showing how the signs of the six functions change as the angle increases from 0° to 360°.

26. Make up a table showing the changes in the numerical values of the functions in each quadrant.

Show by means of a unit circle that

27. $\sin 150° = \sin 30°$.
28. $\cos 210° = -\cos 30°$.
29. $\sin 330° = -\sin 30°$.
30. $\tan 120° = -\tan 60°$.
31. $\cot 240° = \cot 60°$.
32. $\tan 240° = \tan 60°$.

33. If $\sin \phi = \frac{3}{8}$, how many values less than 360° may ϕ have, and in what quadrants do the angles lie? Draw a figure to illustrate your answer.

34. If $\cos \phi = \frac{3}{4}$, how many values less than 360° may ϕ have, and in what quadrants do the angles lie? Draw a figure to illustrate your answer.

35. Considering only angles less than 180°, how many values may ϕ have if $\sin \phi = \frac{1}{8}$? if $\cos \phi = \frac{1}{8}$? if $\cos \phi = -\frac{3}{8}$? if $\tan \phi = \frac{3}{4}$? if $\cot \phi = -8$?

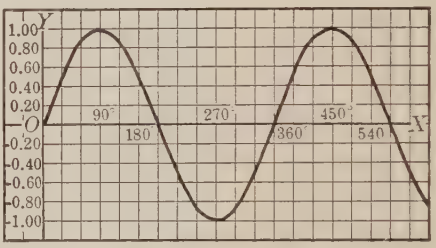
As in Exs. 33 and 34, a figure should be drawn in each of Exs. 35 and 36.

36. Considering angles between 0° and 360°, what are the limits within which ϕ must lie if $\cos \phi = -\frac{3}{5}$? if $\cot \phi = 5$? if $\sec \phi = 20$? if $\csc \phi = -4$?

Graph of sin ϕ. Since $\sin\phi$ is a function of ϕ (page 86), we may draw its graph for various values of ϕ. By the methods given on pages 95–98 we may first find the values of $\sin\phi$ for every 30°, beginning with 0° and going as far as we wish. Computing the values of $\sin\phi$ to the nearest 0.01 in each instance, our table of values would begin as follows:

If $\phi =$	0°	30°	60°	90°	120°	150°	180°	210°	240°	· · ·
$\sin\phi =$	0	0.50	0.87	1.00	0.87	0.50	0	−0.50	−0.87	· · ·

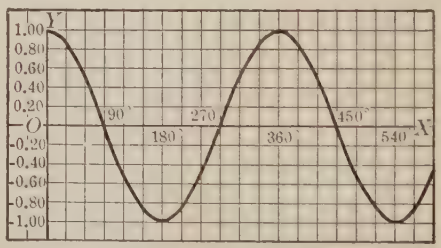
Then, letting each space on the x axis represent 30° and letting each space on the y axis represent 0.20, we may draw the graph here shown.



The graph shows clearly that $\sin\phi$ changes sign in passing through 180°, 360°, 540°, and so on, and that the total range of $\sin\phi$ is from $+1$ to -1.

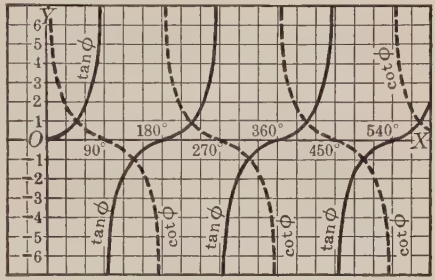
Although we are not at present concerned with negative angles, it is interesting to know that the curve continues in the same way for negative values of ϕ, and that at intervals of 360°, called the *period*, the sine comes back to any given value; that is, $\sin\phi = 1$ when ϕ has the values $-270°$, 90°, 450°, · · · .

Graph of cos ϕ. In the same way we shall find that the graph of $\cos\phi$ has the shape shown in the accompanying figure.



The similarity of this curve to that of $\sin\phi$ above should be noticed. It should also be noticed how much more clearly the facts which were set forth in the table on page 108 stand out when they are presented in graphic instead of tabular form.

Graphs of tan ϕ and cot ϕ. The accompanying figure shows the graphs of tan ϕ and cot ϕ, which are drawn in the same way as the graphs on page 110, except that in this case each space on the y axis represents 1 instead of 0.20.

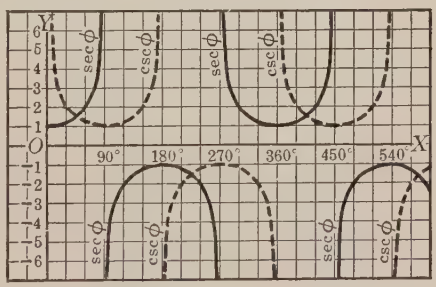


The graph of tan ϕ is shown by the solid line, and that of cot ϕ by the dotted line.

The changes in tan ϕ and cot ϕ, particularly those in which the functions change sign in passing through ∞, are clearly shown, and so are the rapid changes that occur in tan ϕ when ϕ is near 90°, 270°, 450°, $\cdots$, and the similar changes in cot ϕ when ϕ is near 0°, 180°, 360°, $\cdots$.

As with sin ϕ and cos ϕ, the graphs of tan ϕ and cot ϕ continue in the same way for negative values of ϕ.

Graphs of sec ϕ and csc ϕ. In this figure we have the graphs of sec ϕ and csc ϕ, drawn in the same way as those of tan ϕ and cot ϕ above. It is interesting to observe how these functions are related and how clearly the graphs show that the numerical values of both the secant and the cosecant are never less than 1.



The relation between sec ϕ and cos ϕ and that between csc ϕ and sin ϕ can be seen by comparing these graphs with the two on page 110. For example, we see that when $\sin \phi = 0$ we have $\csc \phi = \infty$, and when $\sin \phi = 1$ we have $\csc \phi = 1$. We thus see how the graphs bring out the fact that these functions are the reciprocals of each other.

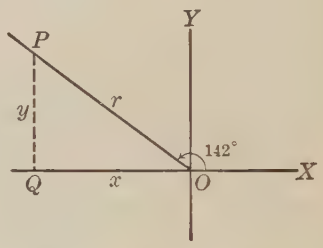
Reduction Formulas. In the tables on pages 362–373 the functions are given for angles from 0° to 90° only. In the solution of practical problems, however, it is often necessary to find the functions of angles greater than 90°.

On pages 95–98, and on page 108, we found that for certain special angles in the second, third, and fourth quadrants the numerical values of the functions were equal to those of certain angles in the first quadrant.

For example, in this figure in which $\angle XOP = 142°$ we can readily see that

$$\angle YOP = 142° - 90° = 52°,$$

and $$\angle POQ = 90° - 52° = 38°.$$



Now, by the definition of sine,

$$\sin XOP = \frac{y}{r},$$

and $$\sin POQ = \frac{y}{r};$$

whence $$\sin XOP = \sin POQ = \sin (180° - 142°) = \sin 38°.$$

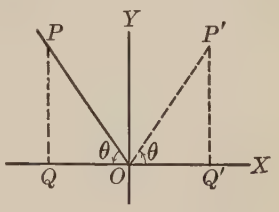
Since there is nothing in the above proof that will not apply equally well to an angle of 125°, 133°, or any other angle in the second quadrant, we see that finding the sine of an angle in this quadrant can be reduced to finding the sine of some angle in the first quadrant.

We shall now develop formulas by which we may reduce the functions of angles from 90° to 360° to functions of angles in the first quadrant. These formulas are very simple, and it is therefore unnecessary to have tables of functions for more than the angles from 0° to 90°.

In considering an angle in the second quadrant, for example, $\angle XOP$ in the figure above, it is evident that we may regard $\angle XOP$ as equal either to $180° - \angle POQ$ or to $90° + \angle YOP$. Thus, in finding sin XOP we may find sin $(180° - \angle POQ)$ or we may find sin $(90° + \angle YOP)$. On the following page we shall develop two sets of formulas, so that in subsequent work we can use whichever set is the more convenient.

Reduction Formulas for the Second Quadrant. If in the first figure below we let $\angle XOP$ represent any angle in the second quadrant, and if we designate $\angle POQ$ as θ, we see that

$$\angle XOP = 180° - \theta.$$



Constructing $\angle XOP'$ in the first quadrant equal to θ and taking OP' equal to OP, we see that $\triangle OPQ$ is congruent to $\triangle OP'Q'$, and hence $OQ = OQ'$ and $QP = Q'P'$.

Then $$\sin(180° - \theta) = \frac{QP}{OP} = \frac{Q'P'}{OP'} = \sin\theta,$$

and $$\cos(180° - \theta) = \frac{OQ}{OP} = -\frac{OQ'}{OP'} = -\cos\theta,$$

where the negative sign is necessary because OQ' is opposite in direction to OQ. Treating the tangent and cotangent in the same way, we have the following formulas:

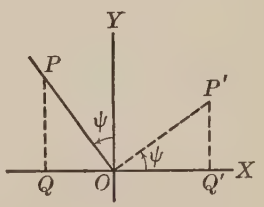
$$\sin(180° - \theta) = \sin\theta \qquad \tan(180° - \theta) = -\tan\theta$$
$$\cos(180° - \theta) = -\cos\theta \qquad \cot(180° - \theta) = -\cot\theta$$

Similarly, designating $\angle YOP$ in this figure as ψ, we have

$$\angle XOP = 90° + \psi.$$

Constructing $\angle XOP'$ equal to ψ and OP' equal to OP, we then have $QP = OQ'$ and $OQ = Q'P'$. Hence

$$\sin(90° + \psi) = \frac{QP}{OP} = \frac{OQ'}{OP'} = \cos\psi,$$



and so on for the other functions of $\angle XOP$. We thus have

$$\sin(90° + \psi) = \cos\psi \qquad \tan(90° + \psi) = -\cot\psi$$
$$\cos(90° + \psi) = -\sin\psi \qquad \cot(90° + \psi) = -\tan\psi$$

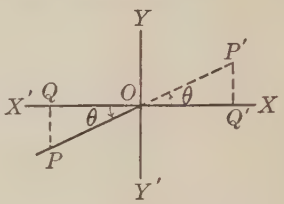
Thus, $\sin 125° = \sin(180° - 55°) = \sin 55°,$

or $\sin 125° = \sin(90° + 35°) = \cos 35°,$

which is correct, because $\sin 55° = \cos(90° - 55°) = \cos 35°$.

Reduction Formulas for the Third Quadrant. If we let $\angle XOP$ in the first figure below represent any angle in the third quadrant, and designate $\angle QOP$ as θ, we see that

$$\angle XOP = 180° + \theta.$$



Constructing $\angle XOP'$ in the first quadrant equal to θ, we see, as on page 113, that

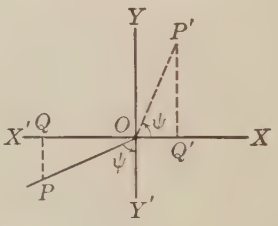
$$\sin(180° + \theta) = -\sin\theta \qquad \tan(180° + \theta) = \tan\theta$$
$$\cos(180° + \theta) = -\cos\theta \qquad \cot(180° + \theta) = \cot\theta$$

If we take OP' equal to OP, we see that $\triangle OQP$ is congruent to $\triangle OQ'P'$, and hence $QP = Q'P'$. Then since $Q'P'$ is opposite in direction to QP (that is, sin XOP is negative and sin θ in the first quadrant is positive), we have

$$\sin(180° + \theta) = \frac{QP}{OP} = -\frac{Q'P'}{OP'} = -\sin\theta.$$

Similarly, if we designate $\angle POY'$ in this figure as ψ, and then construct $\angle XOP'$ in the first quadrant equal to ψ, we have

$$\angle XOP = 270° - \psi.$$



Besides the formulas given above, we thus have the following formulas for reducing the functions of an angle in the third quadrant:

$$\sin(270° - \psi) = -\cos\psi \qquad \tan(270° - \psi) = \cot\psi$$
$$\cos(270° - \psi) = -\sin\psi \qquad \cot(270° - \psi) = \tan\psi$$

From the figure we see that the sign of the line OQ' is opposite to that of its equal, QP, and hence we have

$$\sin(270° - \psi) = \frac{QP}{OP} = -\frac{OQ'}{OP'} = -\cos\psi.$$

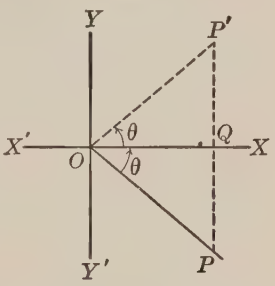
In finding tan 230° we may therefore reduce it to a function in the first quadrant in either of two ways, as follows:

$$\tan 230° = \tan(180° + 50°) = \tan 50°,$$

or

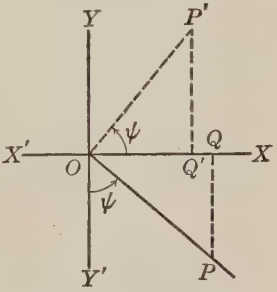
$$\tan 230° = \tan(270° - 40°) = \cot 40°.$$

Reduction Formulas for the Fourth Quadrant. Proceeding in the same way as on pages 113 and 114, we see from this figure that for the fourth quadrant one set of reduction formulas for $\angle XOP$, which we may first designate as $360° - \theta$, is as follows:



$$\sin (360° - \theta) = -\sin \theta$$
$$\cos (360° - \theta) = \cos \theta$$
$$\tan (360° - \theta) = -\tan \theta$$
$$\cot (360° - \theta) = -\cot \theta$$

In the same way the second set of formulas for $\angle XOP$, or $270° + \psi$, is as follows:



$$\sin (270° + \psi) = -\cos \psi$$
$$\cos (270° + \psi) = \sin \psi$$
$$\tan (270° + \psi) = -\cot \psi$$
$$\cot (270° + \psi) = -\tan \psi$$

Thus $\cot 342° = \cot (360° - 18°)$
$= -\cot 18°,$
or $\cot 342° = \cot (270° + 72°)$
$= -\tan 72°.$

General Principle. In these reduction formulas we see that

In using $180° \pm \theta$ or $360° - \theta$ we have the same-named function in the first quadrant; in using $90° + \psi$ or $270° \pm \psi$ we have the co-named function.

It should also be observed that the formulas are true when $\theta > 90°$ or $\psi > 90°$, although these cases are not needed in practical work.

Angles Greater than 360°. In the case of an angle greater than 360°, such as might be generated by a spoke of a rotating wheel, the functions are easily found by remembering that a function of $360° + \phi$, of $720° + \phi$, or, in general, of $n \cdot 360° + \phi$ is the same as the corresponding function of ϕ.

Thus, $\sin 1150° = \sin (3 \cdot 360° + 70°) = \sin 70°.$

Exercises. Reduction Formulas

Express the following functions of angles in the second quadrant as functions of angles in the first quadrant:

1. sin 147°. **4.** cot 167°. **7.** sin 127° 30′.
2. cos 132°. **5.** sec 140°. **8.** cos 143° 49′.
3. tan 151°. **6.** csc 166°. **9.** tan 169° 37′.

Express each of the following as functions of an angle in the first quadrant:

10. sin 196°. **15.** cot 207°. **20.** sin 203° 15′.
11. cos 213°. **16.** sec 273°. **21.** cos 271° 48′.
12. tan 286°. **17.** csc 311°. **22.** tan 296° 50′.
13. cot 198°. **18.** sin 320°. **23.** sec 188° 36′.
14. cos 275°. **19.** cos 218°. **24.** cot 308° 19′.

Express each of the following in two ways as functions of an angle in the first quadrant:

25. sin 101°. **32.** sin 180°. **39.** sin 98° 45′.
26. cos 201°. **33.** cos 270°. **40.** cos 198° 45′.
27. tan 301°. **34.** tan 270°. **41.** tan 498° 48′.
28. cot 401°. **35.** sin 270°. **42.** cot 800° 18′.
29. sin 99°. **36.** sin 400°. **43.** sin 1000° 30′.
30. cos 199°. **37.** csc 400°. **44.** tan 2000° 45′.
31. tan 299°. **38.** cot 400°. **45.** cot 2347° 10′.

46. By showing that the terminal side of the negative angle $-\phi$ coincides with that of the positive angle $360° - \phi$, prove that $\sin(-\phi) = -\sin\phi$, that $\cos(-\phi) = \cos\phi$, that $\tan(-\phi) = -\tan\phi$, and that $\cot(-\phi) = -\cot\phi$.

47. Using the tables on pages 362–367, find the sine, cosine, and tangent of the angle generated by a spoke of a wheel which has revolved counterclockwise through 4068°.

Tables of Natural Functions. On page 100 we learned how to use the four-place tables of natural functions on pages 362–373 in finding the functions of angles given to the nearest 1°. In the nine columns at the right of the column headed **0.0°** (or **0′**) are given the functions for every 0.1°, or 6′, from 0° to 90°.

The practice of using decimal subdivisions of a degree, instead of minutes and seconds, is increasing. Although the older form will probably be used in this country for some time to come, it is desirable that students become familiar with both forms.

Thus, to find sin 37° 42′, we look for **37** in the column at the left and on this line, in the column headed **42′** (or **0.7°**), we find .6115; that is, sin 37° 42′ (or sin 37.7°) = 0.6115.

The heavy-type figures in the tables of tangents, cotangents, secants, and cosecants indicate a change of 1 in the integral part of the function at that point. Thus, on page 367, we see that tan 71° 30′ = 2.9887 and tan 71° 36′ = 3.0061. Similarly, on page 372, csc 14° 24′ = 4.0211 and csc 14′ 30′ = 3.9939; that is, we subtract 1 in this case because the cosecant decreases between 0° and 90°.

Interpolation. On pages 71 and 72 we learned how to use interpolation in connection with the table of logarithms. From the graphs on pages 110 and 111 we see that the same principles can be applied to the functions of an angle. That is, for a very short distance, such as that corresponding to 6′, we may regard the graph of a function of ϕ as a straight line.

Thus, we may say that sin 37° 45′ lies halfway between sin 37° 42′ and sin 37° 48′, the values of which are given in the table, and similarly for the other functions.

The functions themselves do not, of course, increase at the same rate as the angles; that is, sin 90° is not three times sin 30°. In using interpolation for angles differing by less than 6′, however, we are considering a very small change in the angle, and the difference in the rates does not ordinarily affect the fourth decimal place in the function.

Using Interpolation. The following examples illustrate how interpolation is used in the tables of natural functions:

1. Find $\tan 20° 32'$.

Since $\tan 20° 32'$ lies between $\tan 20° 30'$ and $\tan 20° 36'$, we first find from the table of tangents on page 366 that

$$\begin{aligned} \tan 20° 36' &= 0.3759 \\ \tan 20° 30' &= \underline{0.3739} \\ \text{Difference for } 6' &= \underline{0.0020} \end{aligned}$$

The difference for $2'$ is then $\frac{1}{3}$ of this difference; that is, it is $\frac{1}{3} \times 0.0020$, or $0.0006\frac{2}{3}$, which to the nearest 0.0001 is 0.0007. Adding this to $\tan 20° 30'$, because the tangent increases between $0°$ and $90°$, we have

$$\tan 20° 32' = 0.3739 + 0.0007 = 0.3746.$$

2. Find the acute angle for which the cosine is 0.9708.

In the table of cosines on page 364 we locate the position of 0.9708 as lying between 0.9711 and 0.9707 (the cosines of $13° 48'$ and $13° 54'$) which have a total difference of 0.0004. Since 0.9708 is 0.0003 less than 0.9711, this last difference corresponds to $\frac{3}{4}$ of $6'$, or $5'$, to the nearest $1'$. Since the cosine decreases between $0°$ and $90°$, we see that the required angle is $5'$ larger than $13° 48'$; that is, it is $13° 53'$.

Exercises. Using Interpolation

Using the tables on pages 362–373, find

1. $\sin 65° 18'$.	**6.** $\tan 78° 36'$.	**11.** $\tan 64° 10'$.
2. $\sin 65° 24'$.	**7.** $\tan 78° 41'$.	**12.** $\tan 83° 51'$.
3. $\sin 65° 20'$.	**8.** $\cot 45° 15'$.	**13.** $\cot 6° 9'$.
4. $\cos 79° 48'$.	**9.** $\sec 50° 1'$.	**14.** $\sin 45.7°$.
5. $\cos 79° 50'$.	**10.** $\csc 74° 32'$.	**15.** $\sin 45.72°$.

Find the value of ϕ between $0°$ and $90°$ for which

16. $\sin\phi = 0.1685$.	**20.** $\cos\phi = 0.8587$.	**24.** $\csc\phi = 1.0731$.
17. $\sin\phi = 0.1702$.	**21.** $\tan\phi = 3.0287$.	**25.** $\sin\phi = 0.9873$.
18. $\sin\phi = 0.1694$.	**22.** $\cot\phi = 1.9762$.	**26.** $\cot\phi = 6.6930$.
19. $\cos\phi = 0.8590$.	**23.** $\sec\phi = 2.0361$.	**27.** $\tan\phi = 0.2384$.

Difference Columns. To save time and labor in computing differences in the tables of functions, each table has five columns at the right in which the differences for each 1′ are given. The following examples illustrate the use of these difference columns:

1. Find tan 20° 32′.

Instead of proceeding as in Ex. 1 on page 118, we simply find from page 366 that tan 20° 30′ = 0.3739, and then on the same line, in the difference column headed 2′, we find 7. Since the columns are marked "+ Differences," this means that 0.0007 is to be added to tan 20° 30′; that is,

$$\tan 20° 32′ = 0.3739 + 0.0007 = 0.3746.$$

2. Find the acute angle for which the cosine is 0.4104.

On page 365 we locate the position of 0.4104 as following 0.4115, the cosine of 65° 42′. We see that 0.4104 is 0.0011 less than 0.4115, and in difference column 4′ on line 65° we find 11 given as the minus difference for 4′; that is, the required angle is 65° 46′.

3. Find the acute angle for which the tangent is 0.3951.

On page 366 we locate the position of 0.3951 as following 0.3939, the tangent of 21° 30′. We see that 0.3951 is 0.0012 larger than 0.3939, and while we do not find 12 in the difference columns on line 21°, we do find 13 as the plus difference for 4′. Since 12 is nearer 13 than it is 10, the difference for 3′, we see that the required angle is 21° 34′.

It frequently happens that the use of the difference columns will give a result that differs by 1 or 2 in the last decimal place from the result found by interpolation. With a four-place table any result is generally approximate in the fourth figure, and hence such differences may usually be disregarded. When accuracy in the fourth figure is required, a table with a larger number of places should be used.

The method of interpolation, however, must be resorted to in certain cases, such as finding the tangents of angles greater than 75°, as noted in the tables. In the same way, in the second illustrative example on page 118 we find that the difference 3 is given in both columns 4′ and 5′ in the table on page 364. In such cases we therefore resort to interpolation to find the required angle.

Exercises. Using the Tables

Using the columns of differences in the tables on pages 362–373, find

1. sin 7° 15′.	**5.** tan 43° 43′.	**9.** sec 60° 16′.
2. cos 45° 8′.	**6.** tan 68° 11′.	**10.** sec 39° 40′.
3. cos 0° 45′.	**7.** cot 32° 32′.	**11.** csc 20° 49′.
4. sin 69° 38′.	**8.** cot 49° 49′.	**12.** csc 60° 14′.

13. From the roof of a building 71 ft. high the angle of depression of the foot of a tree is found to be 42° 16′. Find the distance from the tree to the building.

14. From a point 135 ft. from a building, the angle of elevation of the top is observed, by means of a transit set 5 ft. above the ground, to be 38° 56′. What is the height of the building?

15. When a flagpole 65 ft. high casts a shadow 40 ft. long, what is the angle of elevation of the sun?

16. When a 40-foot ladder is placed against a building so that it just reaches a window sill 38 ft. 6 in. above the ground, what angle does the ladder make with the ground?

17. The sloping face of a concrete abutment is 27 ft. long and the top is 16 ft. 6 in. higher than the foot. At what angle does the sloping face meet the ground?

18. A tangent 3.2 in. long is drawn from a point P to a circle. If the tangent makes an angle of 50° 39′ at P with a line from P to the center of the circle, what is the length of the radius of the circle?

19. Suppose that you wish to draw two tangents from a point P to a circle 3.4 in. in diameter in such a way that the angle between the tangents shall be 23° 32′. How far from the center of the circle should you take the point P?

Tables of Logarithmic Functions. The computations with trigonometric functions can generally be simplified by using logarithms. Instead of looking up the value of a function in one table and then looking up the logarithm of this value in a table of logarithms, it is much more convenient to have one table which gives the logarithms of the functions, or the *logarithmic functions*, that we need to use.

On pages 374–381 are four-place tables which give the logarithms of the sines, cosines, tangents, and cotangents for every 0.1°, or 6′, from 0 to 90°.

When very small angles or angles very near 90° are used, special tables should be employed, but we shall not need to consider such angles in our present work.

The use of these tables will be evident from the following illustrative examples:

1. Find log sin 16° 32′; that is, find the logarithm of sin 16° 32′.

From page 374,

$$\begin{aligned} \log \sin 16^\circ 36' &= \bar{1}.4559 \\ \log \sin 16^\circ 30' &= \bar{1}.4533 \\ \text{Difference for } 6' &= 0.0026 \end{aligned}$$

Using interpolation, the difference for 2′ is then $\frac{1}{3} \times 0.0026$, or 0.0009, to the nearest 0.0001. Since in the case of sin ϕ the differences are plus (the sine increases between 0° and 90°), we have

$$\sin 16^\circ 32' = \bar{1}.4533 + 0.0009 = \bar{1}.4542.$$

From the difference column headed 2′, we can find at once that 9, or 0.0009, should be added to log sin 16° 30′, and we can thus find log sin 16° 32′ directly from the table. As has been previously stated, however, it is desirable that students learn to use the method of ordinary interpolation so that in examinations, or at other times, they will be able to use tables which are not provided with difference columns.

2. Find the acute angle ϕ for which $\log \tan \phi = \bar{1}.6310$.

On page 378 we locate the position of $\bar{1}.6310$ as following $\bar{1}.6300$, the logarithmic tangent of 23° 6′, and see that $\bar{1}.6310$ is 0.0010 larger than $\bar{1}.6300$. In the difference columns, 10 is given as the plus difference for 3′, and hence the required angle is 23° 9′.

Exercises. Logarithmic Functions

Using interpolation in the tables on pages 374–381, find

1. log sin 15° 15′. **3.** log cos 68° 9′. **5.** log tan 15° 50′.

2. log sin 39° 17′. **4.** log cos 20° 43′. **6.** log cot 52° 38′.

Tables of the logarithmic secants and cosecants are not given, because we rarely need to multiply by sec ϕ or csc ϕ. When we do, we may divide by cos ϕ or sin ϕ instead, since $\sec \phi = \frac{1}{\cos \phi}$ and $\csc \phi = \frac{1}{\sin \phi}$. Since this means simply subtracting a logarithm, or adding a cologarithm if it is easier, tables for log sec ϕ and log csc ϕ are not really needed.

Using the difference columns, find

7. log sin 0° 38′. **10.** log cos 29° 29′. **13.** log tan 60° 33′.

8. log sin 52° 4′. **11.** log tan 15° 50′. **14.** log cot 32° 32′.

9. log cos 98° 16′. **12.** log tan 94° 33′. **15.** log sin 92° 28′.

Find the value of ϕ between 0° and 90° for which

16. log sin $\phi = \bar{1}.9399$. **18.** log tan $\phi = 0.4545$.

17. log cos $\phi = \bar{1}.2980$. **19.** log cot $\phi = \bar{1}.4420$.

Find the value of each of the following, first by using natural functions and then by using logarithms:

20. 16.85 sin 29° 10′. **22.** 157.8 tan 48° 17′.

21. 29.46 cos 63° 25′. **23.** 0.06547 cot 33° 27′.

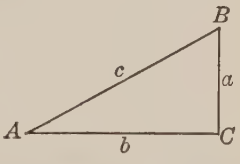
24. In a rectangle 16.38 in. long the diagonal makes an angle of 7° 38′ with the long edge. Find the width and the area of the rectangle.

25. A mechanic has a rectangular steel plate 17.85 in. long and 15.58 in. wide. In sawing the plate into two equal parts, starting at a point on one of the long edges 5.65 in. from a corner, what angle does the saw cut make with the edge?

26. Find the length of the side of the square that can be inscribed in a circle 15.72 in. in diameter.

2. The Right Triangle

Solution of the Right Triangle. In this chapter we have already solved a number of problems which illustrate the practical applications of trigonometry. In each of these problems we had to find either some side or some angle of a right triangle. We shall now consider the more general question of the complete solution of any right triangle when we are given two independent parts of the triangle.



Since $\angle C$ is always a right angle, it cannot be classed as an independent part. Furthermore, $\angle A$ and $\angle B$ are not independent parts, because $\angle A = 90° - \angle B$; that is, each acute angle depends upon the other. Two independent parts must therefore include at least one of the three sides of the triangle.

In speaking of the sides of a right triangle, unless we distinctly mention the three sides, we mean the sides which form the right angle.

There are five possible ways in which two independent parts of a right triangle may be given. The five cases may be stated as follows:

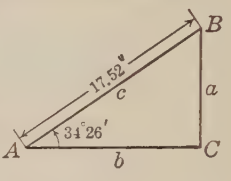
1. Given an acute angle and the hypotenuse.
2. Given an acute angle and the opposite side.
3. Given an acute angle and the adjacent side.
4. Given the hypotenuse and either side.
5. Given the two sides.

Expressed with reference to the standard right triangle, which is lettered as shown above, these five cases are as follows:

1. Given $\angle A$ and c, or $\angle B$ and c.
2. Given $\angle A$ and a, or $\angle B$ and b.
3. Given $\angle A$ and b, or $\angle B$ and a.
4. Given c and a, or c and b.
5. Given a and b.

We shall now consider these five cases in this order and we shall learn how the unknown parts of the triangle can be found in each case.

Case 1. Angle and Hypotenuse. In this case we are given an acute angle and the hypotenuse of the right triangle; for example, solve the right triangle in which $\angle A = 34° 26'$ and $c = 17.52$ in.



That is, given $\angle A$ and c, we are to find $\angle B$, a, and b. Since $\measuredangle A$ and B are not independent, we can find $\angle B$ by a simple subtraction. We therefore proceed as follows:

(1) $B = 90° - A = 90° - 34° 26' = 55° 34'.$

(2) Since $\sin A = \frac{a}{c}$,

we have $a = c \sin A.$

Then
$$\log c = 1.2435$$
$$\log \sin A = \overline{1}.7524$$
$$\log a = 0.9959$$

and $a = 9.906$;

that is, $a = 9.91$ in.

(3) Since $\cos A = \frac{b}{c}$,

we have $b = c \cos A.$

Then
$$\log c = 1.2435$$
$$\log \cos A = \overline{1}.9163$$
$$\log b = 1.1598$$

and $b = 14.45$;

that is, $b = 14.45$ in.

Check. Since $c^2 = a^2 + b^2$,

if we take the squares of the three sides to four figures each, we have

$$c^2 = 17.52^2 = 307.0,$$

and $$a^2 + b^2 = 9.91^2 + 14.45^2 = 98.21 + 208.8 = 307.01.$$

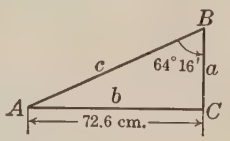
Since the results are approximate, this method of checking is generally only approximate. In this case, however, it shows that the results are correct to four figures.

Exercises. Angle and Hypotenuse

Solve the right triangle in which

1. $\angle A = 28° 17'$, $c = 6.32$ in.
2. $\angle A = 49° 36'$, $c = 72.8$ ft.
3. $\angle A = 58° 52'$, $c = 47.3$ cm.
4. $\angle A = 65° 22'$, $c = 96.2$ yd.
5. $\angle A = 36° 15'$, $c = 101.8$ m.
6. $\angle A = 56° 48'$, $c = 9.78$ in.
7. $\angle A = 72° 26'$, $c = 102.7$ ft.
8. $\angle B = 15° 17'$, $c = 6.8$ m.
9. $\angle B = 29° 42'$, $c = 82.3$ yd.
10. $\angle B = 69°$, $c = 18.75$ mm.
11. $\angle B = 23° 15'$, $c = 125.2$ ft.
12. $\angle B = 46° 18'$, $c = 12.6$ in.
13. $\angle B = 33° 58'$, $c = 425.8$ rd.
14. $\angle B = 78° 25'$, $c = 852.4$ mi.

Case 2. Angle and Opposite Side. In this case we are given an acute angle and the opposite side of the triangle; for example, solve the right triangle in which $\angle B = 64° 16'$ and $b = 72.6$ cm.



That is, given $\angle B$ and b, we are to find $\angle A$, a, and c. Finding $\angle A$ by subtracting $\angle B$ from 90°, as before, we proceed as follows:

(1) $A = 90° - B = 90° - 64° 16' = 25° 44'$.

(2) Since $\cot B = \frac{a}{b}$,

we have $a = b \cot B$.

Then $\log b = 1.8609$

$\log \cot B = \overline{1}.6830$

$\log a = 1.5439$

and $a = 34.99$;

that is, $a = 35.0$ cm.

(3) Since $\sin B = \frac{b}{c}$,

we have $c = \frac{b}{\sin B}$.

Then $\log b = 1.8609$

$\log \sin B = \overline{1}.9546$

$\log c = 1.9063$

and $c = 80.60$;

that is, $c = 80.6$ cm.

Check. Taking the squares of the three sides to four figures, we have

$$c^2 = 80.6^2 = 6496,$$

and $$a^2 + b^2 = 35.0^2 + 72.6^2 = 1225 + 5271 = 6496.$$

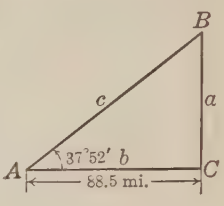
While in this case, as in any other, we might solve by using the natural functions, by using logarithms we save one multiplication by 72.6 and one division by 0.9008.

Exercises. Angle and Opposite Side

Solve the right triangle in which

1. $\angle B = 59°$, $b = 37.0$ ft.
2. $\angle B = 32°$, $b = 4.8$ in.
3. $\angle B = 68° 2'$, $b = 17.8$ yd.
4. $\angle B = 73° 16'$, $b = 20.1$ mi.
5. $\angle B = 62° 45'$, $b = 9.28$ in.
6. $\angle B = 33° 22'$, $b = 10.6$ m.
7. $\angle B = 45° 8'$, $b = 25.6$ cm.
8. $\angle B = 75° 30'$, $b = 56.9$ yd.
9. $\angle A = 19° 8'$, $a = 4.7$ cm.
10. $\angle A = 38° 15'$, $a = 148.4$ ft.
11. $\angle A = 75°$, $a = 9.06$ in.
12. $\angle A = 15° 28'$, $a = 12.6$ mi.
13. $\angle A = 42° 16'$, $a = 96.4$ m.
14. $\angle A = 62° 24'$, $a = 125.0$ rd.
15. $\angle A = 38° 25'$, $a = 64.8$ ft.
16. $\angle A = 56° 20'$, $a = 18.2$ cm.

Case 3. Angle and Adjacent Side. In this case we are given either acute angle of the triangle and the adjacent side; for example, solve the right triangle in which $\angle A = 37° 52'$ and $b = 88.5$ mi.



That is, given $\angle A$ and b, we are to find $\angle B$, a, and c. As in each of the other two cases involving a given acute angle, we can at once find the other acute angle by subtraction. We therefore proceed as follows:

(1) $$B = 90° - A = 90° - 37° 52' = 52° 8'.$$

(2) Since $\tan A = \frac{a}{b}$,

we have $a = b \tan A$.

Then $\log b = 1.9469$

$\log \tan A = \overline{1}.8907$

$\log a = 1.8376$

and $a = 68.80$;

that is, $a = 68.8$ mi.

(3) Since $\cos A = \frac{b}{c}$,

we have $c = \frac{b}{\cos A}$.

Then $\log b = 1.9469$

$\log \cos A = \overline{1}.8973$

$\log c = 2.0496$

and $c = 112.1$;

that is, $c = 112.1$ mi.

Check. Taking the squares of the three sides, as before, we have

$$c^2 = 112.1^2 = 12{,}566,$$

and $$a^2 + b^2 = 68.8^2 + 88.5^2 = 4733 + 7832 = 12{,}565.$$

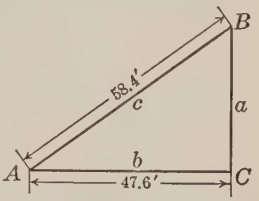
In this case the results, although not the same, are close enough to show that the values of a and c are as nearly accurate as can be expected.

Exercises. Angle and Adjacent Side

Solve the right triangle in which

1. $\angle A = 17°$, $b = 4.32$ in.
2. $\angle A = 42° 8'$, $b = 3.16$ ft.
3. $\angle A = 15° 9'$, $b = 12.8$ cm.
4. $\angle A = 17° 7'$, $b = 3.684$ in.
5. $\angle A = 64° 28'$, $b = 22.8$ mi.
6. $\angle A = 39° 19'$, $b = 125.8$ m.
7. $\angle A = 41.2°$, $b = 285.8$ ft.
8. $\angle B = 5° 2'$, $a = 3.0$ ft.
9. $\angle B = 16° 27'$, $a = 14.37$ m.
10. $\angle B = 39° 19'$, $a = 68.4$ km.
11. $\angle B = 72° 51'$, $a = 140.3$ mi.
12. $\angle B = 45° 28'$, $a = 12.91$ ft.
13. $\angle B = 27° 58'$, $a = 125.2$ rd.
14. $\angle B = 32° 20'$, $a = 416.8$ ft.

Case 4. Hypotenuse and Side. In this case we are given the hypotenuse and either of the two sides of the triangle; for example, solve the right triangle in which $c = 58.4$ ft. and $b = 47.6$ ft.



That is, given c and b, we are to find $\angle A$, $\angle B$, and a. We might, of course, find a from the formula $a = \sqrt{(c+b)(c-b)}$, but the following method of solution is generally simpler:

(1) Since $\cos A = \frac{b}{c}$,

we have $\log b = 2.6776 - 1$
and $\log c = 1.7664$
Then $\log \cos A = 0.9112 - 1$
$= \bar{1}.9112$,
and $A = 35° 24'$.

(2) $B = 90° - 35° 24' = 54° 36'$.

(3) Since $\tan A = \frac{a}{b}$,

we have $a = b \tan A$.
Then $\log b = 1.6776$
$\log \tan A = \bar{1}.8517$
$\log a = 1.5293$
and $a = 33.83$;
that is, $a = 33.8$ ft.

Check. Taking the squares of the three sides to four figures, we have

$$c^2 = 58.4^2 = 3411,$$

and $$a^2 + b^2 = 33.8^2 + 47.6^2 = 1142 + 2266 = 3408.$$

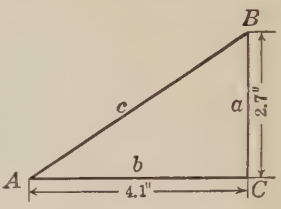
As on page 126, the results here are not exactly the same, but since they agree to three figures, they give a fair check.

Exercises. Hypotenuse and Side

Solve the right triangle in which

1. $c = 8.0$ ft., $b = 6.0$ ft.
2. $c = 26\frac{1}{2}$ yd., $b = 13\frac{1}{4}$ yd.
3. $c = 8.2$ cm., $b = 6.8$ cm.
4. $c = 11.2$ mi., $b = 8.4$ mi.
5. $c = 7\frac{1}{4}$ in., $b = 3\frac{1}{2}$ in.
6. $c = 28.6$ in., $b = 13.5$ in.
7. $c = 169.7$ ft., $b = 107.3$ ft.
8. $c = 516\frac{1}{2}$ mi., $b = 432\frac{1}{2}$ mi.
9. $c = 7.0$ in., $a = 5.0$ in.
10. $c = 23.5$ in., $a = 17.2$ in.
11. $c = 19.6$ cm., $a = 14.8$ cm.
12. $c = 198.1$ m., $a = 143.7$ m.
13. $c = 16.24$ rd., $a = 8.16$ rd.
14. $c = 253.8$ mi., $a = 116.9$ mi.
15. $c = 62.24$ in., $a = 58.01$ in.
16. $c = 17.65$ ft., $a = 11.99$ ft.

Case 5. Two Sides. In the last case we are given the two sides of the right triangle; for example, solve the right triangle in which $a = 2.7$ in. and $b = 4.1$ in.



That is, given a and b, we are required to find $\angle A$, $\angle B$, and c. As in Case 4, it is evident that if we compute one acute angle, we can find the other by subtraction. We therefore proceed as follows:

(1) Since $\tan A = \frac{a}{b}$,

we have $\log a = 1.4313 - 1$

and $\log b = 0.6128$

Then $\log \tan A = 0.8185 - 1$

$= \bar{1}.8185,$

and $A = 33° 22'.$

(2) $B = 90° - 33° 22' = 56° 38'.$

(3) Since $\sin A = \frac{a}{c}$,

we have $c = \frac{a}{\sin A}.$

Then $\log a = 0.4313$

$\log \sin A = \bar{1}.7404$

$\log c = 0.6909$

and $c = 4.908;$

that is, $c = 4.9$ in.

Check. Taking the squares of the three sides, as before, we have

$$c^2 = 4.9^2 = 24.01,$$

and $$a^2 + b^2 = 2.7^2 + 4.1^2 = 7.29 + 16.81 = 24.10.$$

Exercises. Two Sides

Solve the right triangle in which

1. $a = 16$ in., $b = 17$ in.
2. $a = 3.4$ in., $b = 3.4$ in.
3. $a = 49$ ft., $b = 36$ ft.
4. $a = 53$ yd., $b = 75$ yd.
5. $a = 26$ mi., $b = 18$ mi.
6. $a = 34$ rd., $b = 43$ rd.
7. $a = 16.8$ ft., $b = 22.5$ ft.
8. $a = 39.2$ m., $b = 28.4$ m.
9. $a = 12.6$ in., $b = 12.6$ in.
10. $a = 8.24$ m., $b = 6.35$ m.
11. $b = 16.2$ mi., $a = 8.5$ mi.
12. $a = 32.5$ in., $b = 26.7$ in.
13. $a = 18.5$ yd., $b = 23.0$ yd.
14. $b = 2.18$ in., $a = 8.12$ in.
15. $b = 36.0$ ft., $a = 28.5$ ft.
16. $a = 192.0$ rd., $b = 45.2$ rd.
17. $a = 170.8$ mi., $b = 304.2$ mi.
18. $a = 462$ km., $b = 578$ km.
19. $a = 14.96$ m., $b = 13.87$ m.
20. $a = 736.9$ ft., $b = 800.2$ ft.

Area of a Right Triangle. Since the area of any triangle is half the product of the base and the height, we see that in a right triangle with the sides a and b the area T is given by the formula

$$T = \tfrac{1}{2}\, ab.$$

The letter T is here used for "area" in place of the customary A in order to avoid confusion with $\angle A$ of the triangle.

From this formula we can find the area of a right triangle if a and b are given or if they can be found from the parts which are given.

For example, when given either acute angle and the hypotenuse, as in Case 1 on page 124, we can easily find the area of the triangle.

That is, after finding $\log a$ and $\log b$, as on page 124, we have

$$\begin{aligned} \log a &= 0.9959 \\ \log b &= 1.1598 \\ \operatorname{colog} 2 &= \overline{1}.6990 \\ \log T &= \overline{1}.8547 \end{aligned}$$

Since antilog $1.8547 = 71.57$, the area is therefore 71.57 sq. in.

The following solutions show how the area may be found in each of the other four cases on pages 125–128:

CASE 2.
$$\begin{aligned} \log a &= 1.5439 \\ \log b &= 1.8609 \\ \operatorname{colog} 2 &= \overline{1}.6990 \\ \log T &= \overline{3}.1038 \\ T &= 1270. \end{aligned}$$
1270 sq. cm. *Ans.*

CASE 3.
$$\begin{aligned} \log a &= 1.8376 \\ \log b &= 1.9469 \\ \operatorname{colog} 2 &= \overline{1}.6990 \\ \log T &= \overline{3}.4835 \\ T &= 3044. \end{aligned}$$
3044 sq. mi. *Ans.*

CASE 4.
$$\begin{aligned} \log a &= 1.5293 \\ \log b &= 1.6776 \\ \operatorname{colog} 2 &= \overline{1}.6990 \\ \log T &= 2.9059 \\ T &= 805.2. \end{aligned}$$
805.2 sq. ft. *Ans.*

CASE 5.
$$\begin{aligned} \log a &= 0.4313 \\ \log b &= 0.6128 \\ \operatorname{colog} 2 &= \overline{1}.6990 \\ \log T &= 0.7431 \\ T &= 5.535. \end{aligned}$$
5.5 sq. in. *Ans.*

In the last case we might have taken simply $\frac{1}{2} \times 2.7 \times 4.1$, but since we have already found the logarithms of a and b, the above method of solution is generally easier.

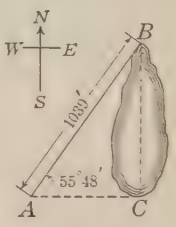
Exercises. Right Triangles

1. From the top of a lighthouse which rises vertically 95 ft. out of the water, the angle of depression of a rock is found to be 5° 13′. How far is the rock from the lighthouse?

In each case the student should draw a simple diagram to illustrate the problem.

2. From a point 30 ft. above the water, on the deck of a steamer, the angle of elevation of the light in a lighthouse is found to be 0° 52′. If the light is 85 ft. above the level of the water, how far is the steamer from the lighthouse?

3. To find the length of a pond, a surveyor set his transit at C, exactly south of a tree B, and sighted a line to the west. He then set a stake at A on this line, moved the transit to A, and found that $\angle A = 55° 48'$. He measured AB and found it to be 1039 ft. What is the length of the pond?



4. If from a point 150 ft. from a tree the angle of elevation of the top is 25.1°, what is the height of the tree?

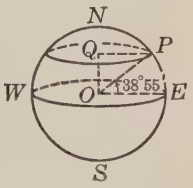
5. A ladder 40 ft. long rests against a wall and makes an angle of 74° 5′ with the ground. What angle does the ladder make with the building? How high does it reach on the wall? How far is the foot of the ladder from the wall?

6. What is the gradient (the angle of slope) of a road that rises uniformly 1 ft. in every 65 ft. measured along the road? 1 ft. in every 65 ft. measured horizontally?

7. A rectangle 18.5 in. long has a diagonal that is 20.3 in. long. What angle does the diagonal make with the base? with the side adjacent to the base?

8. A ship is sailing northeast at a speed of 12 mi./hr. At what rate is the ship moving due east? due north?

9. Taking the earth as a sphere with a radius of 4000 mi., find the length of the radius of the circle of latitude of Washington, D. C., which is located in 38° 55′ north latitude.

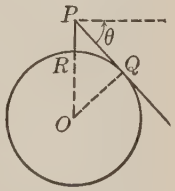


10. In Ex. 9 find the length of 1° of longitude at Washington.

This length is $\frac{1}{360}$ of the circumference of the circle of latitude.

11. In Ex. 9 find the difference between the length of 1° of latitude and the length of 1° of longitude at Washington.

12. From a height of 13,200 ft. above the sea, what is the angle of depression of the horizon?



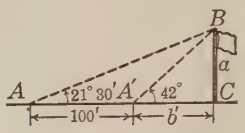
From Ex. 9 we have $OQ = OR = 4000$ mi., and hence OQ and OP are known. With a four-place table we shall find that $\angle OPQ$ is so near 90° that we can only determine two limits between which the angle lies. In the figure, RP is greatly exaggerated, so that $\angle OPQ$ has not its proper size.

13. A rectangular field 850 ft. long and 520 ft. wide is divided into two equal parts by a straight fence that starts at a point on one of the long sides 110 ft. from a corner. How long is the fence that divides the field? What angle does the fence make with the long sides?

14. If from any point on the edge of a circular pond in a park the angle of elevation of the top of a 70-foot flagpole rising from the center of the pond is 38° 20′, what is the diameter of the pond?

15. From a point on a bridge 30 ft. above a stream the angle of depression of a wooden block that is floating steadily downstream is observed to be 5° 30′ at a certain instant. If it then takes the block 1 min. 40 sec. to reach the bridge, what is the rate in miles per hour at which the stream is flowing?

16. From a point A on the ground the angle of elevation of the top of a flagpole is found to be 21° 30′, while from a point A', which is 100 ft. nearer the pole in a straight line, the angle of elevation of the top is 42°. What is the height of the flagpole?



In the figure we see that $a = b' \tan 42°$ and $a = (b' + 100) \tan 21° 30'$. From these two equations we can eliminate a and thus find b'. It is then easy to find a by using either $\tan A$ or $\tan A'$.

17. From a point offshore the angle of elevation of the top of a cliff is observed to be 2° 18′. After sailing $\frac{1}{2}$ mi. in a straight line toward the cliff, the angle of elevation is then 3° 25′. What is the height of the cliff in feet?

18. From the same point on the ground the angles of elevation of the top and bottom of a flagstaff on top of a building 50 ft. high are 44° 50′ and 29° 30′ respectively. What is the height of the flagstaff?

19. Find the radii of the circles inscribed in and circumscribed about a regular decagon 1.3 in. on a side.

20. Find the area of a regular octagon 3 in. on a side.

21. Find the area of the circle circumscribed about a regular pentagon with a perimeter of 7.4 in.

22. Find the difference between the areas of a regular hexagon and a regular octagon, each with a perimeter of 32 in.

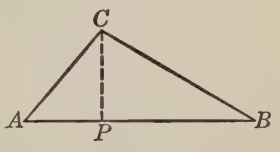
23. Find the difference between the perimeters of a regular pentagon and a regular hexagon, each of which has an area of 24 sq. in.

24. Find the perimeter of a regular polygon of 100 sides inscribed in a circle 1 in. in diameter.

The perimeter of the polygon will be very near the circumference of the circle, which can be found from the formula $C = \pi d$. Since $d = 1$ in this case, the result should be an approximate value for π.

3. The Oblique Triangle

The Oblique Triangle. It is evident that we may divide any oblique triangle, such as the $\triangle ABC$ here shown, into two right triangles, which can then be solved by the methods that we have already learned. Although we shall use this method in developing formulas for an oblique triangle, it is much more convenient in practice to use the formulas thus developed than it is to divide the triangle into right triangles every time that we wish to solve it.



Cases to be Considered. From geometry we know that a triangle is determined if the following parts are given:

1. Two angles and any side.
2. The three sides.
3. Two sides and the included angle.
4. Two sides and the angle opposite one of them.

The fourth case, it will be recalled, is known as "the ambiguous case," since there are generally two triangles which satisfy the given conditions; that is in this case the triangle is not, in general, determined.

For example, suppose that in the above figure we are given $\angle B$ and the sides AC and BC, and wish to construct the triangle. The figure, as given, is evidently one solution of the problem. If, however, we let CA swing counterclockwise about C as a center, we see that A will again fall on PB between P and B. The triangle thus formed will have $\angle B$ and the sides AC and BC, as required, and is therefore a second solution.

The four cases will now be considered, and in discussing the fourth one we shall take up the special cases in which there are two, one, or no solutions of the problem.

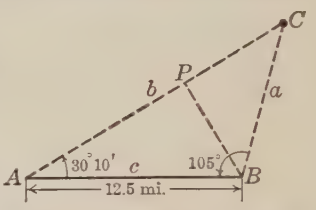
In taking up the first two cases we shall first show how the triangle can be solved by dividing it into right triangles, and we shall then show how such a solution may be shortened.

Case 1. Two Angles and a Side. Since the sum of the three angles of a triangle is 180°, the third angle can be found when the sizes of two of the angles are known.

For example, if a triangle has two angles of 47° 15′ and 121° 7′ respectively, the third angle contains $180° - (47° 15' + 121° 7')$, or 11° 38′.

The problem of solving a triangle when two angles and any side are known therefore reduces to solving a triangle when two angles and the included side are known.

For example, in locating the position of a forest fire at a point C, the observer at station A finds that the line of sight to C makes an angle of 30° 10′ with a line running to the station B, 12.5 mi. away. The observer at B finds that the line of sight to C makes an angle of 105° with AB. How far is the fire from B?



Drawing BP in the above figure perpendicular to AC, we see that

$$BP = c \sin A.$$

Then
$$\log 12.5 = 1.0969$$
$$\log \sin 30° 10' = \overline{1}.7012$$
$$\log BP = 0.7981$$

Since
$$\sin C = \frac{BP}{a},$$

we have
$$a = \frac{BP}{\sin C},$$

so that by finding the size of $\angle C$ we can find a. We therefore find that

$$C = 180° - (30° 10' + 105°) = 180° - 135° 10' = 44° 50'.$$

Then, as above,
$$\log BP = 0.7981$$
$$\log \sin 44° 50' = \overline{1}.8483$$
$$\log a = 0.9498$$

Hence
$$a = 8.908.$$

That is, the fire is 8.9 mi. from B, which is the nearest station to it.

Although in practice the position of C is located by plotting AC and BC upon a map from their bearings, the above example shows the principles involved in cases of this type.

Exercises. Two Angles and a Side

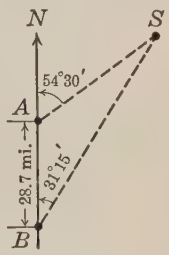
1. On page 134 find the distance from the station A to C.

We see that $AP = BP \cot A$ and that $PC = BP \cot C$.

2. In the $\triangle ABC$ in which $AB = 46.2$ in., $\angle A = 69°$, and $\angle B = 37° 41'$ what is the length of the side b?

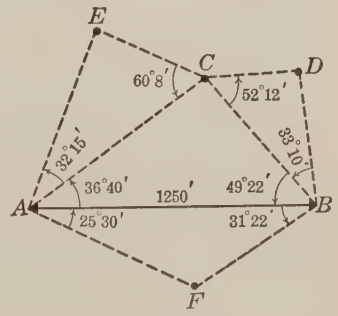
3. In the $\square ABCD$ in which $\angle B = 120° 15'$, $AB = 8.72$ in., and the diagonal AC makes an angle of $26° 30'$ with AB, what are the lengths of the side BC and the diagonal AC?

4. The principles used in locating the position of a ship at sea by radio compass involve this case of an oblique triangle. By means of a direction finder the ship's officers are able to determine the directions from which radio signals sent out by the shore stations at A and B are coming. For example, suppose that B is 28.7 mi. south of A, and that the officers find that $\angle SAN = 54° 30'$ and $\angle SBA = 31° 15'$. What is the distance of the ship from A? from B?



As in the case on page 134, the position of S is located on a map by the bearings, and the latitude and longitude are thus determined.

5. In making a survey of an area of any size, surveyors generally locate the important points by means of triangles laid off from a carefully-measured base line. The sides of the triangles on the base line are then computed and serve as bases for other triangles. In this figure, in which AB is a 1250-foot base line, find the length of BF; of BD; of CE.



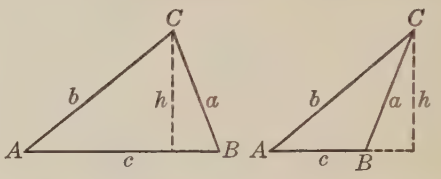
6. In Ex. 5 find the area inclosed by the four triangles.

Law of Sines. If in either of the two triangles shown below we draw the altitude h from C, we see that in both cases

$$\frac{h}{b} = \sin A.$$

In the first case, also,

$$\frac{h}{a} = \sin B,$$



and in the second case, where h is outside $\triangle ABC$, we have

$$\frac{h}{a} = \sin (180° - B) = \sin B.$$

Dividing $\frac{h}{b} = \sin A$ by $\frac{h}{a} = \sin B$, we then have

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B}.$$

If we draw the altitude from the vertex A, it can be shown in the same way that

$$\frac{b}{c} = \frac{\sin B}{\sin C}, \quad \text{or} \quad \frac{b}{\sin B} = \frac{c}{\sin C}.$$

We therefore have the following important relation, which is known as the *Law of Sines:*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

That is, since we have shown that the first and third ratios are each equal to the second, it follows that all three are equal.

This law may also be expressed as follows:

In a triangle the ratios of the sides to the sines of the opposite angles are equal.

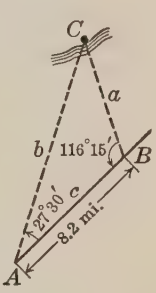
From the Law of Sines we see that in the problem on page 134 we could have found at once that

$$a = \frac{c \sin A}{\sin C},$$

and from this relation we could then have easily found the length of a without first finding the length of BP.

Law of Sines Applied. The advantages of using the Law of Sines in the solution of an oblique triangle may be illustrated by the following problem:

From a ship sailing on the course AB in this figure it is found that when the ship is at A the line of sight to a lighthouse C makes an angle of 27° 30′ with AB. After sailing 8.2 mi. to B, the angle to the lighthouse becomes 116° 15′. How far is the position of the ship from the lighthouse at each observation?



Since the side c of the triangle is given, we shall need to use $\sin C$ in finding the other two sides. We therefore find first that

$$C = 180° - (27° 30' + 116° 15') = 180° - 143° 45' = 36° 15'.$$

Using the Law of Sines, we may then proceed as follows:

(1) From $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$,

we have $a = \dfrac{c \sin A}{\sin C}$.

Then	$\log c = 0.9138$
	$\log \sin A = \bar{1}.6644$
	$\operatorname{colog} \sin C = \underline{0.2282}$
	$\log a = 0.8064$
Hence	$a = 6.403.$

(2) From $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$,

we have $b = \dfrac{c \sin B}{\sin C}$.

Then	$\log c = 0.9138$
	$\log \sin B = \bar{1}.9527$
	$\operatorname{colog} \sin C = \underline{0.2282}$
	$\log b = 1.0947$
Hence	$b = 12.44.$

That is, at A the ship is 12.4 mi. from the lighthouse, and at B it is 6.4 mi. distant.

In finding $\log \sin B$, or $\log \sin 116° 15'$, we know from page 113 that $\sin (90° + \psi) = \cos \psi$ and $\sin (180° - \theta) = \sin \theta$. We may therefore look up $\log \cos (116° 15' - 90°)$, or $\log \cos 26° 15'$, or, if we prefer, we may look up $\log \sin (180° - 116° 15')$, or $\log \sin 63° 45'$. The two results are, of course, the same since $\cos \phi = \sin (90° - \phi)$.

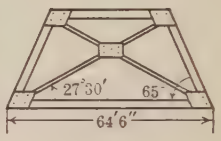
Since the Law of Sines was developed by dividing an oblique triangle into two right triangles, the above work simply combines the solutions of two right triangles. A comparison of the solution for a, above, with the work on page 134 will show the saving in computation.

Exercises. Law of Sines

1. The front wall of a building that stands on a side hill makes an angle of 108° 16′ with the ground. From a point 92 ft. downhill from the building, measured along the slope, the line of sight to the top makes an angle of 24° 10′ with the slope of the hill. What is the height of the building?

2. From one end of a base line 1250 ft. long the line of sight to a distant landmark makes an angle of 36° 28′ with the base line, and from the other end the angle is 112° 32′. How far is the landmark from each end of the base line?

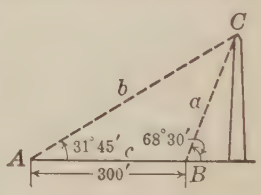
3. In a bridge truss which has the shape shown in this figure the sloping ends make an angle of 65° with the base, and the diagonal braces make an angle of 27° 30′ with the base. Find the lengths of the ends, the top, and the diagonals of a truss 64 ft. 6 in. long.



4. From a point at the foot of a mountain the angle of elevation of the top is 43° 20′. From a point $\frac{1}{2}$ mi. up a trail that rises at a uniform slope of 32° the angle of elevation of the top is 49° 38′. Find the height of the mountain in feet.

The angle of elevation in each case is measured from the horizontal.

5. From a point A the angle of elevation of the top of a monument is found to be 31° 45′, while from the point B, 300 ft. nearer the monument, the angle is 68° 30′. Find the height of the monument.

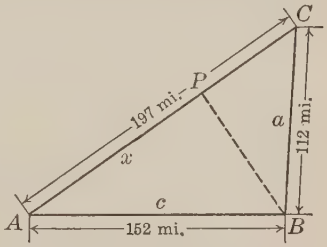


By using the Law of Sines we can find a in the figure; we can then apply Case 1, page 124.

6. From Ex. 5 develop a formula for finding the height h of an object in terms of the angles of elevation α and β taken from two points in line with the object and at the distance d from each other.

Case 2. Three Sides. Although the second case, — that of solving the triangle when the three sides are given, — is less important in practical work than the first one, it has considerable importance in the study of the triangle, since from it we can develop another law similar to the Law of Sines.

As in the case on page 134, we shall first solve this case by using two right triangles. For example, suppose that three flying-fields, A, B, and C, are so situated that A is 152 mi. from B and 197 mi. from C, while B and C are 112 mi. apart. What is the difference in the courses taken by two planes one of which is flying from A to B and the other from A to C?



Drawing the altitude BP, and letting x be the length of AP, we have

$$\cos A = \frac{x}{c}.$$

If we can find the value of x, we can solve the problem; that is, from $\cos A$ we can find $\angle A$, as required. In $\triangle ABP$ we see that

$$\overline{BP}^2 = 152^2 - x^2,$$

and in $\triangle BCP$ $$\overline{BP}^2 = 112^2 - (197 - x)^2.$$

Hence $$152^2 - x^2 = 112^2 - 197^2 + 394\,x - x^2,$$

or $$394\,x = 197^2 + 152^2 - 112^2;$$

whence $$x = 125.302, \text{ approximately.}$$

Then $$\cos A = \frac{125.302}{152} = 0.8244, \text{ approximately;}$$

whence $$A = 34^\circ\,28'.$$

In practice, each course would be given as the "angle magnetic"; that is, as the angle measured clockwise from the north-and-south line through A. The angular difference between the courses, however, is always $\angle A$.

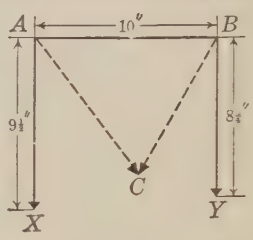
For example, the course AB might be 87° 32′, magnetic, while AC might be 53° 4′, but the angular difference would be 34° 28′.

In the above case we could find $\angle C$ in a similar way, and then find $\angle B$ by subtracting the sum of $\angle A$ and C from 180°, thus completing the solution of the triangle.

Exercises. Three Sides

1. In a motor-boat race the first leg of the course is 4.1 mi. long, the second leg between the turning stakes is 6.2 mi. long, and the last leg runs 4.7 mi. straight to the starting-point. Find the angles between the legs of the course.

2. Two pendulums, one $9\frac{1}{2}$ in. long and the other $8\frac{3}{4}$ in., are hung from points 10 in. apart. Through how many degrees of arc must each pendulum swing so that their ends shall just touch?



The required angle for the first pendulum is evidently twice $\angle XAC$, and to find this angle we must first find $\angle CAB$.

Find the angles of a triangle in which the sides are as follows:

3. 10 in., 10 in., 10 in.
4. 6 ft., 8 ft., 10 ft.
5. 52 in., 64 in., 21 in.
6. 76 ft., 98 ft., 37 ft.
7. 42 mi., 56 mi., 64 mi.
8. 46 cm., 56 cm., 82 cm.
9. 9 km., 10 km., 12 km.
10. 8.8 m., 8.8 m., 10.7 m.

11. Three cities, A, B, and C, are so situated that B is 159 mi. due east of A and 176 mi. from C, while C is 84 mi. from A. In what direction is C from A? What two answers are admissible? Explain your answers.

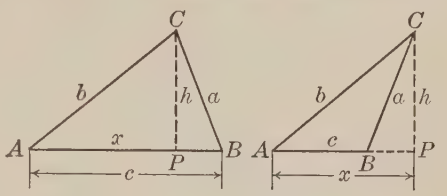
12. If in a quadrilateral $ABCD$, $AB = 3$ in., $BC = 4$ in., $CD = 4$ in., $DA = 5$ in., and $AC = 6$ in., what is the size of each angle of the quadrilateral?

13. While flying in a fog over a city A towards a city C, 54 mi. away, an aviator loses his way. When the fog lifts he finds himself over a city B, which is 87 mi. from A and 38 mi. from C. How many degrees must he change his course in order to reach C?

The required angle is evidently $180° - \angle CBA$.

Law of Cosines. If in either of the triangles here shown we draw the altitude CP, or h, and designate the length of AP as x, we see that in both cases

$$\cos A = \frac{x}{b}.$$



In the first case, also,

$$h^2 = b^2 - x^2,$$

while in the second case, where h is outside $\triangle ABC$, we have

$$h^2 = a^2 - (x - c)^2.$$

Equating these two values of h^2, we then have

$$b^2 - x^2 = a^2 - x^2 + 2cx - c^2;$$

whence

$$x = \frac{b^2 + c^2 - a^2}{2c}.$$

Substituting this value of x in the expression for $\cos A$, we then have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

in which $\cos A$ is expressed in terms of the three sides of $\triangle ABC$. In the same way we can show that

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

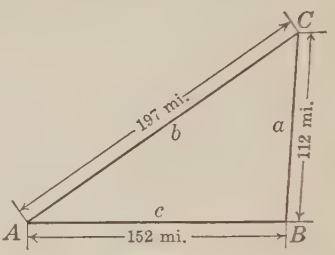
and

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

These relations are known as the *Law of Cosines*. Just as the Law of Sines was used in shortening the solution of a triangle when two angles and a side were known, so the Law of Cosines can be used when the three sides are known.

The student should observe that we need only learn the law for $\cos A$, since it may be written for the other angles by making cyclic changes in the letters. That is, in writing the law for $\cos B$ we simply put b for a, c for b, and a for c.

Law of Cosines Applied. In order to see how the solution of a triangle by means of the Law of Cosines is related to the solution given on page 139, let us again consider the problem there given. Briefly stated, we are given the $\triangle ABC$ in which $a = 112$, $b = 197$, and $c = 152$, all measurements being made in miles. The problem requires us to solve this triangle for $\angle A$.



By using the Law of Cosines we can shorten the solution as follows:

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{197^2 + 152^2 - 112^2}{2 \cdot 197 \cdot 152}$$

$$= \frac{38{,}809 + 23{,}104 - 12{,}544}{59{,}888}$$

$$= 0.8244, \text{ approximately;}$$

whence $A = 34° 28'$.

On account of the plus and minus signs it is not convenient to use logarithms when solving by the Law of Cosines. This was also true in the solution given on page 139. There are, however, various formulas in which logarithms can be used in solving a triangle when the three sides are given. For example, we have the formula

$$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}},$$

in which s represents the semiperimeter of the triangle; that is, $s = \frac{1}{2}(a + b + c)$.

The development of formulas like this requires more time than can be allowed in a course of this nature. The above formula may be used, however, if it is so desired, and on page 143 it is shown in the exercises how formulas for $\cos \frac{1}{2} B$ and $\cos \frac{1}{2} C$ may be written.

If it is desired to find $\angle B$ in the above case, it can be readily found by using the Law of Cosines for $\cos B$, and then $\angle C$ can be found from the relation $\angle C = 180° - (\angle A + \angle B)$, thus completing the solution of the triangle.

Exercises. Law of Cosines

1. From the formula for $\cos \frac{1}{2} A$ on page 142 derive formulas for $\cos \frac{1}{2} B$ and $\cos \frac{1}{2} C$ by cyclic changes of letters.

2. In the $\triangle ABC$ shown on page 142 use the formulas developed in Ex. 1, above, in finding $\angle B$; $\angle C$.

3. From two points, A and B, which are $\frac{3}{4}$ mi. apart on a main highway, two branch roads run to a point C that is 2560 ft. from A and 2730 ft. from B. At what angle do the two branch roads meet?

4. Two targets 3 mi. apart are respectively 3068 yd. and 3479 yd. from a coast-defense gun. When the gun is trained on one target, through what angle must it be turned to fire on the other target?

5. The corners A and B of a triangular lot ABC are located by stone markers, but the marker at C has been lost. The deed to the property shows that the side AB is 354.8 ft. long, BC is 127.2 ft., and AC is 382.6 ft. At what angle with AB should a surveyor run a line on which C can be located by measuring from A? Check the solution by finding $\angle B$ and using $\angle A$, $\angle B$, and AC to find BC.

6. From the Law of Cosines develop a formula for a in terms of b, c, and $\cos A$. Show that when $\angle A = 90°$ this formula becomes the Pythagorean Theorem.

Since $\angle A = 90°$ is a special case of the Law of Cosines, this law is sometimes called the *Generalized Pythagorean Theorem.*

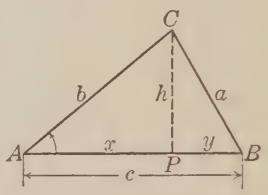
7. From the Law of Cosines prove that the square of the side opposite an acute angle of a triangle is equal to the sum of the squares of the other two sides minus twice the product of one of these sides and the projection of the other side upon it.

8. By means of the Law of Cosines prove that an equilateral triangle is also equiangular.

Case 3. Two Sides and the Included Angle. The third case mentioned on page 133 is that in which we are given two sides and the included angle, from which we are to find the third side and the other two angles.

There are various methods of treating this case, and each one requires more computation than the first two cases that we have considered. Before solving a problem that will illustrate this case, we shall first consider the general method of attack.

For example, suppose that in this figure we know the lengths of b and c and the size of $\angle A$. Let us first draw the altitude h from C so as to divide the triangle into two right triangles. Then designating AP as x and PB as y, we have



$$x = b \cos A,$$

and hence we can find the length of x. We can now find the length of y by subtracting x from c.

Then in $\triangle APC$, which is a right triangle, we have

$$h = b \sin A,$$

and hence we can readily find the length of the altitude h.

Since $$\tan B = \frac{h}{y},$$

we can now find the size of $\angle B$. It is then a simple matter to find the size of $\angle C$ by using the relation

$$\angle C = 180° - (\angle A + \angle B).$$

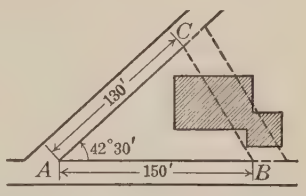
From $$\sin B = \frac{h}{a},$$

we now have $$a = \frac{h}{\sin B},$$

and hence we can find the length of a and thus complete the solution of the triangle.

From the above outline it is obvious that a large amount of computation is involved in such a solution.

Practical Use of Case 3. In developing an old estate at the corner of two intersecting streets, it is planned to put in the new street shown by the dotted lines in this figure, and thus leave a small triangular park ABC after the old house is torn down. If $\angle A$ is found to be 42° 30′, and it is planned to make AB 150 ft. long and AC 130 ft., what will be the length of BC, and at what angle at B should the new street be laid out?

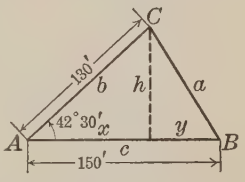


Drawing a working figure, we have the $\triangle ABC$ here shown, in which $b = 130$, $c = 150$, and $\angle A = 42° 30'$. Letting the altitude h from C divide AB into the segments x and y, we have

$$x = b \cos A = 130 \times 0.7373$$
$$= 95.85, \text{ approximately,}$$

and $$y = 150 - 95.85 = 54.15.$$

Also, $$h = b \sin A = 130 \times 0.6756$$
$$= 87.83, \text{ approximately.}$$



We may now complete the solution by the two following computations:

(1) Since $\tan B = \dfrac{h}{y}$,

we have
$$\log h = 1.9436,$$
$$\log y = \underline{1.7336}$$
$$\log \tan B = \overline{0.2100}$$

Hence $B = 58° 20'$.

(2) Since $a = \dfrac{h}{\sin B}$,

we have
$$\log h = 1.9436$$
$$\log \sin B = \overline{1.9300}$$
$$\log a = 2.0136$$

Hence $a = 103.2$.

That is, the third side a, or BC, is 103.2 ft. long, and $\angle B = 58° 20'$.

Another Method of Solution. From the Law of Cosines,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc},$$

we have
$$a^2 = b^2 + c^2 - 2\,bc \cos A,$$

which might have been used in finding the length of a.

The computation would have been laborious, since logarithms could not be conveniently used, and hence the above method is generally better in ordinary cases such as we shall consider in our work.

A Third Solution of Case 3. In the practical solution of the third case of the oblique triangle, the relation

$$\frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$$

and the corresponding forms involving $\tan\frac{1}{2}(B+C)$ and $\tan\frac{1}{2}(C+A)$, which can be obtained by making cyclic changes in the letters, are often used.

As already stated, we have not the time to develop all these relations, but they may be used in the exercises if desired.

In the problem on page 145 we know that $\angle B+\angle C=180^\circ-\angle A$. Hence from

$$\tan\tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\cdot\tan\tfrac{1}{2}(B+C),$$

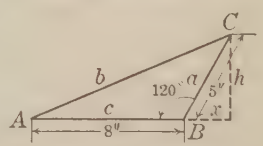
in which we know b, c, and $B+C$, we can find the value of $B-C$ by using logarithms. Then from $B+C$ and $B-C$ we can find the size of $\angle B$, and can then find the length of a by the method on page 137.

Exercises. Two Sides and Included Angle

1. From the top of a tower the angle included between the lines of sight to two landmarks known to be 6.5 mi. and 8.2 mi. respectively from the tower is found to be 65° 22′. What is the distance between the two landmarks?

2. Two railroad trains leave a station at the same time on tracks which diverge at an angle of 38°. If one train travels 35 mi./hr. and the other 45 mi./hr., how far apart are the trains at the end of 1 hr.?

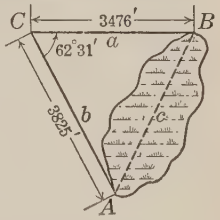
3. In the $\triangle ABC$ shown below, $AB=8$ in., $BC=5$ in., and $\angle B=120^\circ$. Find the sizes of $\measuredangle A$ and C and the length of b, by using (1) the first method on page 145, (2) the Law of Cosines, and (3) the tangent relation above.



In (1) we see that $\tan A=\dfrac{h}{c+x}$, in which $h=a\sin(180^\circ-B)$ and $x=a\cos(180^\circ-B)$. In (3), by writing c for a and b for c, the tangent relation becomes $\dfrac{c+a}{c-a}=\dfrac{\tan\frac{1}{2}(C+A)}{\tan\frac{1}{2}(C-A)}$.

4. Two airplanes leave a flying-field at the same time. The first plane follows the course 38° 24′, magnetic, at a speed of 152 mi./hr., while the second takes the course 67° 40′, magnetic, at a speed of 148 mi./hr. How far apart are the planes at the end of $1\frac{1}{2}$ hr.?

5. In finding the distance AB across the swamp shown in this figure, a surveyor measures AC, BC, and $\angle C$, and finds that $AC = 3825$ ft., $BC = 3476$ ft., and $\angle C = 62° 31'$. Find the length of AB.



In this case it will be more convenient to choose a method that will permit the use of logarithms throughout the solution.

6. In response to an S O S call from a ship at A, another ship at B, 175 mi. due east of A, starts toward A at a speed of 12 mi./hr. At the same time a third ship at C, which is 186 mi. from B in a direction bearing 3° 15′ west of north, also starts for A at a speed of 16 mi./hr. Which ship will reach A first, and how long will it take?

7. From an east-and-west automobile highway a branch highway bears 22° 30′ west of north. At a point 5.6 mi. east of the road fork a dirt road runs through the woods to a point on the branch highway 4.8 mi. from the fork. How many miles could a motorist save by taking the dirt road instead of following the main road through the fork?

8. Find the length of the diagonal of a regular pentagon that is 5.24 in. on a side.

The sum S of the interior angles of a regular polygon of n sides is given by the formula $S = (n - 2) \cdot 180°$.

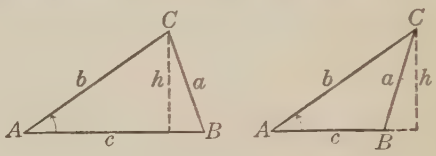
9. In a parallelogram in which the lengths of the adjacent sides are 8.2 in. and 5.6 in. and the included angle is 68° 15′, find the length of each diagonal and the angles made by the diagonals with the base and with each other.

Case 4. The Ambiguous Case. On page 133 it was stated that the fourth case, the one in which we are given two sides of the triangle and the angle opposite one of them, is what is known in geometry as "the ambiguous case"; that is, if we have a, b, and $\angle A$ given, the $\triangle ABC$ may be either one of the two triangles shown in the figure below.

Let us now examine this case in connection with the Law of Sines, which states that

$$\frac{a}{\sin A} = \frac{b}{\sin B},$$

or $$\sin B = \frac{b \sin A}{a}.$$



Since $\sin (180° - B) = \sin B$, when we find $\angle B$ by means of the above formula we cannot tell whether it is acute or obtuse. Similarly, since $\angle C = 180° - (\angle A + \angle B)$ there are two possible values for $\angle C$.

From the above figure we can also see that

If $\angle A$ is acute and the value of a lies between the values of $b \sin A$ and b, there are two solutions.

That is, since $h = b \sin A$ in either triangle, a must be greater than h, or $b \sin A$, and less than b if there are to be two triangles.

If $a < b \sin A$, or if $\angle A$ is obtuse and $a \geqq b$, there is no solution.

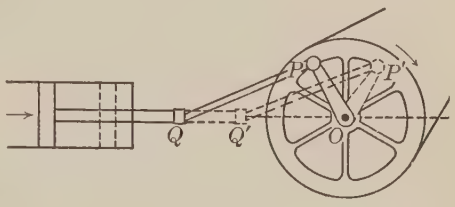
If $a < h$, it is evident that a cannot intersect c. If $\angle A$ is obtuse and $a \geqq b$ (a is equal to or less than b), then either $\angle B = \angle A$ or $\angle B > \angle A$, that is, both $\angle A$ and B are obtuse, which is impossible.

In each of the other cases there is one solution.

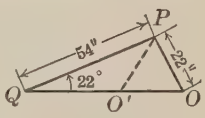
The number of solutions can often be determined by inspection, or by considering the value of $\log \sin B$. If $\log \sin B = 0$, then $\sin B = 1$ and $B = 90°$; that is, $\triangle ABC$ is a right triangle. If $\log \sin B > 0$, then $\sin B > 1$, which is impossible (page 107); that is, there is no solution in this case. If $\log \sin B < 0$, there is one solution if $a \geqq b$, and two solutions if $a < b$.

Solving the Ambiguous Case. The following problems illustrate typical solutions of this case of the oblique triangle:

1. In this figure, which shows the working parts of a steam engine, suppose that the connecting rod PQ is 54 in. long and the crank OP is 22 in. long, all measurements being made from centers. When PQ makes an angle of 22° with the horizontal, what angle does OP make with the horizontal?



That is, as shown in the figure below, we are given a triangle in which $QP = 54$ in., $OP = 22$ in., and $\angle Q = 22°$. From the Law of Sines we have



$$\frac{OP}{\sin Q} = \frac{PQ}{\sin O},$$

or $$\sin O = \frac{PQ \sin Q}{OP} = \frac{54 \times 0.3746}{22} = 0.9195, \text{ approximately.}$$

Hence $$\angle O = 66° 51',$$

or $$\angle O = 180° - 66° 51' = 113° 9';$$

that is, there are two solutions, as we saw from the first figure.

2. Solve the $\triangle ABC$ in which $a = 1.5$ in., $b = 1.9$ in., and $\angle A = 108°$.

In this case $a < b$ and $\angle A > 90°$. Hence $\angle B > \angle A$, which is not possible, since a triangle cannot have two obtuse angles.

3. Solve the $\triangle ABC$ in which $a = 3.6$ in., $b = 8.6$ in., and $\angle A = 30°$.

Since $b \sin A = 8.6 \times 0.5000 = 4.3$, we see that $a < b \sin A$, and hence the triangle is impossible, as shown on page 148.

4. Solve the $\triangle ABC$ in which $a = 3$ in., $b = 6$ in., and $\angle A = 30°$.

In this case $b \sin A = 6 \times 0.5000 = 3 = a$, and hence $\sin B = 1$; that is, $\angle B = 90°$. Hence $\triangle ABC$ is a right triangle, which is easily solved.

Exercises. The Ambiguous Case

1. If the crank and connecting rod of an engine like the one illustrated on page 149 are respectively 18 in. and 38 in. long, what angle does the crank make with the rod when the latter makes an angle of 13° with the horizontal?

2. In Ex. 1 what is the largest angle that the connecting rod can make with the horizontal?

3. From a point P, which is 9.7 in. from the center O of a circle 5.2 in. in diameter, a line is drawn making an angle of 11° 30′ with OP. What is the length of this line between P and the point where it cuts the circle?

4. From an observatory at A a landmark C, 6.8 mi. away, is found to bear 11° 12′ east of north, while a landmark B, known to be 4.6 mi. from C, bears 43° 38′ east of north from A. How far is it from A to B?

5. When 18.2 mi. south of a reef that rises just above the surface of the water, a steamer changes its course to N. 12° 30′ E. If the distance of the horizon visible from the bridge of the steamer is 9.8 mi., for how many miles will the reef be visible from the steamer?

N. 12° 30′ E. means a direction bearing 12° 30′ east of north.

6. In the $\triangle ABC$ in which $AB = 5.4$ in. and $\angle A = 28° 30'$, a circle 6 in. in diameter is drawn with the vertex B as center. Find the length of the chord cut off by AC.

7. Without drawing a figure to scale state whether or not it is possible to construct the $\triangle ABC$ when given that

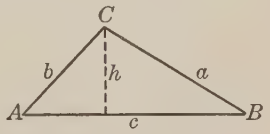
(1) $AC = 20$ in., $BC = 15$ in., and $\angle A = 36°$;
(2) $AC = 8$ in., $BC = 11$ in., and $\angle A = 35°$;
(3) $AC = 16$ in., $BC = 8$ in., and $\angle A = 30°$;
(4) $AC = 11$ in., $BC = 8$ in., and $\angle A = 108°$.

State the number of solutions in each case and give reasons.

Area of a Triangle. We have seen that we can always solve a triangle when three independent parts are given, with the exception of the possible double solution in the ambiguous case (the case in which two sides and the angle opposite one of them are given). Consequently, if we are given any three independent parts of a triangle, we can always find two sides and the included angle. Every practical case of finding the area of a triangle can therefore be reduced to the one in which two sides and the included angle are given, and hence we need to consider only this case of finding the area.

Thus, if b, c, and $\angle A$ are known in this triangle, we see that if we draw the altitude h from C, and represent the area of the triangle by T, we have

$$T = \tfrac{1}{2}\, ch.$$



Then, since $h = b \sin A$,

we have $T = \tfrac{1}{2}\, bc \sin A.$

As on page 129, the letter T is here used for "area" in place of the conventional A in order to avoid confusion with $\angle A$.

For example, suppose that in measuring a triangular city lot a surveyor finds that the sides which include an angle of 62° 40′ are 58.7 ft. and 69.2 ft. respectively. What is the area of the lot?

That is, we are given a triangle, like $\triangle ABC$ above, in which $b = 58.7$, $c = 69.2$, and $\angle A = 62° 40'$, and are required to find the area. Using logarithms in the above formula for T, we have

$$\begin{aligned} \log b &= 1.7686 \\ \log c &= 1.8401 \\ \log \sin A &= \bar{1}.9486 \\ \operatorname{colog} 2 &= \underline{\bar{1}.6990} \\ \log T &= 3.2563 \end{aligned}$$

whence $T = 1804.$

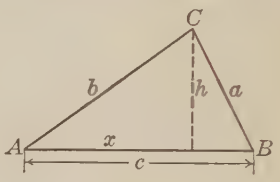
That is, the area of the lot is 1804 sq. ft. Since we used four-place tables, the last figure of the result may be only approximate, and in practice in such a case larger tables would be used.

Heron's Formula. In practical work it is often most convenient to measure the three sides of a triangle. The area T may then be computed by the formula

$$T = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semiperimeter of the triangle with the sides a, b, and c; that is, $s = \frac{1}{2}(a+b+c)$. This formula, which bears the name of the mathematician Heron of Alexandria, is easily proved by the Law of Cosines.

For example, in this figure if we take $\cos^2 A$ to mean the square of $\cos A$, we have



$$h^2 = b^2 - x^2 = b^2 - b^2\cos^2 A,$$

and by the Law of Cosines

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$

We therefore have

$$\begin{aligned} h^2 &= b^2 - b^2\left(\frac{b^2+c^2-a^2}{2\,bc}\right)^2 \\ &= \left(b + b\cdot\frac{b^2+c^2-a^2}{2\,bc}\right)\left(b - b\cdot\frac{b^2+c^2-a^2}{2\,bc}\right) \\ &= \left(\frac{2\,bc + b^2 + c^2 - a^2}{2\,c}\right)\left(\frac{2\,bc - b^2 - c^2 + a^2}{2\,c}\right) \\ &= \frac{(b+c)^2 - a^2}{2\,c}\cdot\frac{a^2 - (b-c)^2}{2\,c} \\ &= \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{4\,c^2}. \end{aligned}$$

If we now let $a+b+c = 2\,s$, we have

$$h^2 = \frac{2\,s\cdot 2(s-a)\cdot 2(s-c)\cdot 2(s-b)}{4\,c^2}.$$

Then $$h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)};$$

whence $$T = \tfrac{1}{2}\,ch = \sqrt{s(s-a)(s-b)(s-c)}.$$

After finding s, $s-a$, $s-b$, and $s-c$, we can use logarithms.

Exercises. Area of a Triangle

1. Find the area of an equilateral triangle that is 7.2 in. on a side, (1) by using the geometric formula $A = \frac{1}{4} s^2\sqrt{3}$, where s is the side; (2) by using the formula on page 151; (3) by using Heron's formula.

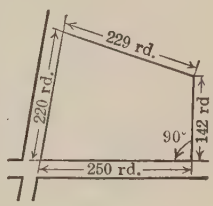
2. Find the length of a diagonal and the area of a regular pentagon that is 3.8 in. on a side.

3. Find the area of a regular polygon of seven sides that are each 9.3 in. long.

4. In measuring a triangular city lot a surveyor finds that the sides are 354.2 ft., 416.8 ft., and 624.5 ft. What is the area of the lot?

5. In laying out a regular octagon for a design a draftsman first draws a circle with a radius of 3.8 in., and in the circle he then inscribes the octagon. Find the area of the design.

6. A farm situated at a crossroad has the shape and dimensions shown in this figure, which was sketched from the description in the deed to the property. How many acres are there in the farm?



There are 160 sq. rd. in 1 A. (acre).

7. From a steel plate that weighs 9 lb./sq. ft. a workman cuts out a triangular plate of which the three sides are respectively 24.2 in., 38.5 in., and 45.0 in. What is the weight of the triangular plate?

8. The angle formed at the ridgepole of an A-shaped barn roof is 105°, and the width of the barn is 45 ft. Find the area of the triangular gable at each end of the barn.

9. Show that the area of a parallelogram is half the product of its diagonals and the sine of the acute angle included between the diagonals.

Exercises. Review

1. Show that the area of the square circumscribed about a circle is $33\frac{1}{3}\%$ greater than that of a regular polygon of twelve sides inscribed in the circle.

2. A regular hexagon is inscribed in a circle with a radius of 7.2 in. What is the area of that part of the circle which is not covered by the hexagon? The area of the hexagon is what part of the area of the circle?

3. A regular five-pointed star is inscribed in a circle 5 in. in diameter. What is the area of the star? of the part of the circle not covered by the star?

4. In Ex. 3 a regular pentagon may be formed by joining the points of the star. The area of this pentagon is how many times that of the pentagon formed by the dotted lines at the center?

5. A circular hole with a radius of $4\frac{1}{2}$ in. is cut in the center of a regular hexagonal plate that is 9 in. on a side. What is the area of the rest of the plate?

6. The area of the regular hexagon formed by joining the midpoints of the sides of a regular hexagon is what part of the area of the larger hexagon?

7. A surveyor wishes to measure the area of a piece of property which has the shape shown in this figure. Draw a diagram to show what lines and angles you would measure.

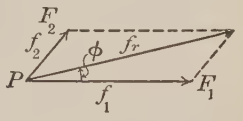
The surveyor would probably lay off perpendicular axes through some point in the field, but we do not need to consider this method.

8. A circle 9.2 in. in diameter is cut by two parallel chords that are respectively 6 in. and 7 in. long. What is the area of the part included between the two chords?

Consider all possible solutions.

4. Problems from Mechanics and Physics

Parallelogram of Forces. In the study of physics we learn that if two forces act in different directions from a point, as indicated by the lines PF_1 and PF_2 here shown, and if the magnitudes of these forces are represented by the lengths f_1 and f_2 respectively, the resultant of the two forces is represented in magnitude and direction by the diagonal f_r of the parallelogram in which f_1 and f_2 are adjacent sides. The substance of this relation is known as the *Law of the Parallelogram of Forces.*



For example, if two forces of 50 lb. and 20 lb. act at an angle of 50° to each other, what is the magnitude and direction of the resultant?

That is, in the above figure we have $f_1 = 50$, $f_2 = 20$, $\angle F_1PF_2 = 50°$, and $\angle F_1 = 180° - 50° = 130°$. Then, by the Law of Cosines,

$$f_r^2 = f_1^2 + f_2^2 - 2f_1f_2 \cos 130° = 50^2 + 20^2 - 2 \cdot 50 \cdot 20 \cdot (-0.6428) = 4185.6;$$

whence $f_r = 64.70$, approximately.

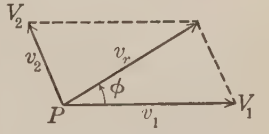
In finding the direction ϕ of the resultant, we have

$$\cos \phi = \frac{f_r^2 + f_1^2 - f_2^2}{2 f_r f_1} = \frac{4185.6 + 2500 - 400}{2 \cdot 64.70 \cdot 50} = 0.9715, \text{ approximately;}$$

whence $\phi = 13° 42'$.

That is, the resultant is a force of 64.7 lb. acting at an angle of 13° 42′ with the 50-pound force.

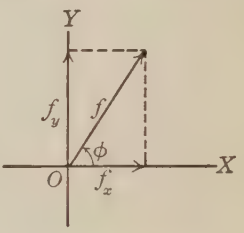
Parallelogram of Velocities. We have a similar law known as the *Law of the Parallelogram of Velocities* for finding the resultant of two given velocities. Thus, if v_1 and v_2 in this figure represent the magnitudes of two velocities in the directions PV_1 and PV_2 from P, the resultant velocity is represented in magnitude and direction by the diagonal v_r of the parallelogram of which v_1 and v_2 are adjacent sides.



Horizontal and Vertical Components. If the magnitude and direction of a force are known, we can readily determine the horizontal and vertical forces, known as the *horizontal and vertical components*, of which the given force is a resultant.

For example, if a force of magnitude f is acting in the direction ϕ from the horizontal, we see from this figure that

$$\cos\phi = \frac{f_x}{f} \quad \text{and} \quad \sin\phi = \frac{f_y}{f}.$$



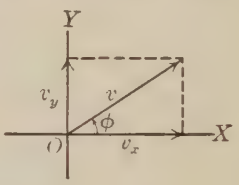
For the horizontal and vertical components f_x and f_y we then have

$$f_x = f\cos\phi$$

and

$$f_y = f\sin\phi.$$

In the same way, the formulas for the horizontal component v_x and the vertical component v_y of a velocity of magnitude v acting in the direction ϕ from the horizontal are respectively



$$v_x = v\cos\phi$$

and

$$v_y = v\sin\phi.$$

Exercises. Resultants and Components

Find the horizontal and vertical components when

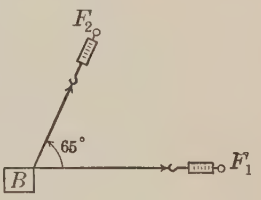
1. $f = 625$, $\phi = 62° 33'$.
2. $f = 2200$, $\phi = 34° 56'$.
3. $f = 2685$, $\phi = 95° 15'$.
4. $v = 28.9$, $\phi = 36° 20'$.
5. $v = 36.7$, $\phi = 67° 18'$.
6. $v = 164.2$, $\phi = 102° 45'$.

7. When $v_x = 36$ and $v_y = 24$, find the direction from the horizontal and the magnitude of the resultant velocity.

8. When two forces of 625.7 lb. and 832.5 lb. act at an angle of $124° 30'$ to each other, what are the direction and the magnitude of the resultant?

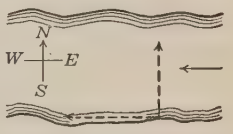
Exercises. Miscellaneous Problems

1. When two spring balances are attached to a heavy block B, as here shown, the block starts to move when the balances F_1 and F_2 indicate 3.8 lb. and 2.6 lb. respectively. What single force must be applied to the block to produce the same effect as the two forces, and in what direction from the first force (F_1) must it be applied?



2. In Ex. 1 what are the horizontal and vertical components of the resultant force; that is, what forces acting at right angles to each other would produce the same effect?

3. While the wind is blowing from the south at 3 mi./hr., a boy starts a toy sailboat from the southern bank of a river that flows westward at 4 mi./hr. If the boat continues to sail before the wind, how far will it travel in 5 min., and in what direction from due north?



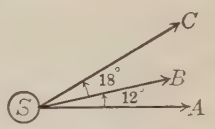
4. A steamer is traveling on a course 11° 30′ east of north, while the wind is blowing directly from the east. If the velocities of the ship and the wind are equal, in what direction will a weather vane on the ship point?

5. If while a steamer is traveling due north at 16 mi./hr., the wind, which is blowing from the east, causes the smoke from the steamer to float off in a direction 30° west of south, what is the velocity of the wind?

6. Solve Ex. 5 when the steamer is traveling northeast at the same rate, and a wind from the east causes the smoke to float off in a direction 60° west of south.

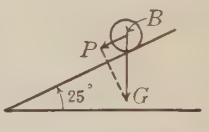
7. When a ship is traveling on the course N. 68° 30′ E. at a speed of 18 mi./hr., at what velocity is it moving due north? due east?

8. In pulling a stump three ropes are attached to it as shown in this figure. If a man at A exerts a force of 200 lb. on his rope, one at B exerts a force of 180 lb., and one at C a force of 225 lb., what is the magnitude of the resulting pull on the stump? In what direction from SA is this pull exerted?



First find the magnitude and the direction of the resultant of the two forces applied at A and B, and then find the resultant of this force and the force at C.

9. If a ball weighing $6\frac{1}{2}$ lb. is placed on a plank inclined to the horizontal at an angle of 25°, what force must be exerted in a direction parallel to the plank to keep the ball from rolling down the plank?



The weight of the ball, $6\frac{1}{2}$ lb., acts downward as shown by BG in the figure. Since the ball must roll down the plank, however, only the component BP parallel to the plank is effective in causing the ball to roll. Hence, to keep the ball from rolling, an equal force must be applied in the opposite direction. Since BG is perpendicular to the horizontal and BP is parallel to the plank, what is the size of $\angle PBG$?

The component of a force in any direction is the product of the force and the cosine of the angle between it and the component.

10. An automobile weighing 2800 lb. is standing on a paved road that rises 5 ft. in every 100 ft. measured horizontally. What force is required of the brakes of the car to keep the car from rolling downhill?

11. A body falling from rest will fall approximately 16 ft. the first second. How far will a ball roll down a plane inclined at an angle of 20° to the horizontal in the same time?

12. In Ex. 11 find the distance when the plane is inclined at an angle of 50°; at an angle of 75°.

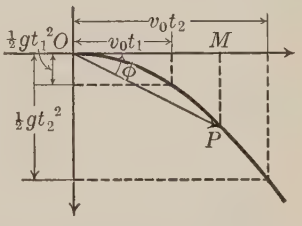
13. In Ex. 11 the body will fall approximately 64 ft. in 2 sec. How far will the ball roll in the same time?

14. A picture weighing $8\frac{1}{2}$ lb. is hung by a wire over a hook in such a way that the two parts make an angle of 60°. What is the tension upon each part of the wire?

In this case we have two equal forces acting at 60° to each other, and their resultant is equal and opposite to the weight of the picture.

15. Solve Ex. 14 when the angle is 20°; 135°; 175°. What angle is best for hanging a heavy picture? Why?

16. Disregarding the resistance of the air, an object projected horizontally from a point O above the ground at an initial velocity of v_0 feet per second would travel horizontally a distance of v_0t feet in t seconds were it not for the action of gravity. Gravity causes the object to drop during its flight in such a way that the distance dropped in t seconds is $\frac{1}{2}gt^2$, where g, the acceleration of gravity, is 32.16 ft./sec. By locating points for various values of t it can be shown that the path of the object is the curve shown in the figure. The resultant drawn to any point P on this curve therefore shows the direction (ϕ) and the distance (OP) of the object from its starting-point. Find OP and ϕ when $v_0 = 50$ ft./sec. and $t = 1\frac{1}{2}$ sec.; when $v_0 = 50$ ft./sec. and $t = 10$ sec.



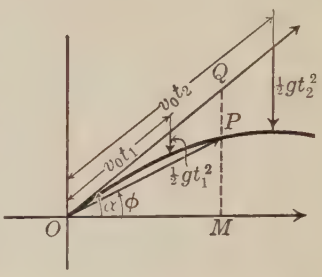
We have $\tan\phi = \dfrac{MP}{OM} = \dfrac{\frac{1}{2}gt^2}{v_0t}$, from which we can find the size of ϕ.

In this exercise it is more convenient to take the positive direction of the y axis as downward.

17. In Ex. 16 find the time in which ϕ would be 60° for an object with an initial velocity of 2000 ft./sec.

18. If a bullet is fired horizontally from the top of a tower 100 ft. high with an initial velocity of 2000 ft./sec., how long will it take the bullet to reach the ground, and how far from the foot of the tower will it strike?

19. Disregarding the resistance of the air, as in Ex. 16, if an object is projected upward from the point O at an angle (α in the figure) with the horizontal, its path is the curve here shown. When the object is at the point P, we have $OM = v_0 t \cos\alpha$ and $MP = MQ - PQ = v_0 t \sin\alpha - \frac{1}{2} gt^2$. Hence we can find the horizontal and vertical distances reached by the object at any point in its flight. Find these distances when $v_0 = 2000$ ft./sec., $t = 5$, and $\alpha = 36°$.

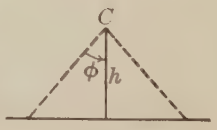


20. In Ex. 19 find ϕ and OP, which give the direction and distance of the object from its starting-point.

21. When the object in Ex. 19 strikes the ground, we have $MP = 0$. Solving the equation $v_0 t \sin\alpha - \frac{1}{2} gt^2 = 0$ for t, we have $t = 0$ or $t = \dfrac{2\, v_0 \sin\alpha}{g}$. Since $t = 0$ at O, the second value of t is evidently the time of flight of the object. Find this time when $v_0 = 2000$ ft./sec. and $\alpha = 40°$.

22. Substituting the second value of t given in Ex. 21, we see that when the object strikes the ground we have $OM = \dfrac{2\, v_0^2 \sin\alpha}{g} \cos\alpha$, which is the horizontal distance from O, called the "range." Find the range in Exs. 19 and 21.

23. The intensity of illumination (I) transmitted to a horizontal surface by a light of candle power C placed h feet above the surface is given by the formula $I = \dfrac{C \cos^3\phi}{h^2}$, where ϕ, as shown in the figure, is the angle that the rays make with the vertical. Find the intensity of the illumination produced by a 300 candle-power light placed 5 ft. above a horizontal surface and so focused that $\phi = 40°$.



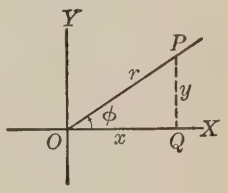
5. Analytic Trigonometry

Analytic Trigonometry. Thus far we have considered trigonometry chiefly from the numerical point of view; that is, we have learned its value in practical cases involving some kind of measurement. The theoretical, or analytic, part of trigonometry is concerned with the relations of the functions, independent of numerical considerations. Although these relations are often helpful in developing formulas for practical use, in a course like this they are of less importance.

For example, as we have already seen, a study of the functions in the first two quadrants gave us the formula $\sin(180° - \theta) = \sin\theta$, from which we can find the sine of an obtuse angle from a table of sines of acute angles.

Trigonometric Identities. A large part of analytic trigonometry is concerned with proving or developing identities, such, for example, as

$$\tan\phi = \sin\phi \sec\phi.$$



Since $\frac{y}{x} = \tan\phi$, $\frac{y}{r} = \sin\phi$, and $\frac{r}{x} = \sec\phi$,

and since $$\frac{y}{x} = \frac{y}{r} \cdot \frac{r}{x},$$

we see that $\tan\phi = \sin\phi \sec\phi$.

Trigonometric Equations. In analytic trigonometry we also need to know how to solve trigonometric equations. For example, suppose that we wish to find a value for ϕ in the first quadrant that will satisfy the equation

$$\cot\phi = \tfrac{1}{3}\tan\phi.$$

Since $\cot\phi = \dfrac{1}{\tan\phi}$, we substitute this value for $\cot\phi$ and have

$$\frac{1}{\tan\phi} = \frac{\tan\phi}{3};$$

whence $$\tan^2\phi = 3,$$

and $$\tan\phi = \sqrt{3} = 1.7321.$$

Looking up this value of $\tan\phi$ in the tables, we then have

$$\phi = 60°.$$

Exercises. Analytic Trigonometry

Prove each of the following identities:

1. $\sin^2\phi + \cos^2\phi = 1.$
2. $\cot\phi \sin\phi = \cos\phi.$
3. $\tan\phi = \dfrac{\sin\phi}{\cos\phi}.$
4. $\sin\phi = \pm\dfrac{\tan\phi}{\sqrt{1+\tan^2\phi}}.$
5. $1 + \tan^2\phi = \sec^2\phi.$
6. $1 + \cot^2\phi = \csc^2\phi.$
7. $\tan\phi = \dfrac{\sin\phi}{\sqrt{1-\sin^2\phi}}.$
8. $\cos\phi = \pm\dfrac{\cot\phi}{\sqrt{1+\cot^2\phi}}.$

Find the value of ϕ in each of the following equations:

9. $2\cos\phi = \sec\phi.$
10. $4\sin\phi = \csc\phi.$
11. $\sin^2\phi = 3\cos^2\phi.$
12. $2\sin^2\phi + \cos^2\phi = \frac{3}{2}.$
13. $3\tan^2\phi - \sec^2\phi = 1.$
14. $\tan\phi = 2\sin\phi.$
15. $\sec\phi = \tan\phi \cdot \sqrt{2}.$
16. $\sin^2\phi - \cos\phi = \frac{1}{4}.$
17. $\tan^2\phi - \sec\phi = 1.$
18. $\tan^2\phi + \csc^2\phi = 3.$

19. It is easily proved in more advanced trigonometry that $\sin(\phi + \phi') = \sin\phi\cos\phi' + \cos\phi\sin\phi'$. From this relation develop a formula for $\sin 2\phi$.

The formulas given in Exs. 19 and 20 are to be assumed.

20. As in Ex. 19, it is proved in more advanced courses that $\cos(\phi + \phi') = \cos\phi\cos\phi' - \sin\phi\sin\phi'$. From this relation develop a formula for $\cos 2\phi$.

From the relations given in Exs. 19 and 20 prove that

21. $\tan(\phi + \phi') = \dfrac{\tan\phi + \tan\phi'}{1 - \tan\phi\tan\phi'}.$

22. $\tan 2\phi = \dfrac{2\tan\phi}{1 - \tan^2\phi}.$

23. $\cot(\phi + \phi') = \dfrac{\cot\phi\cot\phi' - 1}{\cot\phi' + \cot\phi}.$

24. $\cot 2\phi = \dfrac{\cot^2\phi - 1}{2\cot\phi}.$

CHAPTER V

ANALYTIC GEOMETRY

1. General Nature of the Work

Relation to Graphs. In our study of graphs (pages 25–44) we found that it was often helpful to draw a figure to represent a functional relation between two quantities. For example, in studying the formula $A = \pi r^2$ on page 28, we drew the graph there shown, which is simply a representation by a curve of the relation that exists between A and r in the formula. To a certain extent we may therefore look upon a graph as a picture of a formula, and upon a formula as a description of a graph.

Analytic geometry is chiefly concerned with a study of the geometric forms that represent certain algebraic expressions. We shall find in some cases, as in the one mentioned above, that it is more convenient to study the curve that represents the expression, and in other cases we shall find it more convenient to study the algebraic expression which is represented by the curve.

In our study of graphs and of trigonometry we also found that it was convenient to use certain technical terms, such as *axes, origin, abscissa, ordinate, coördinates,* and *quadrants,* in discussing graphs or the location of points and angles. In this and the following chapters we shall make an even greater use of these terms.

In this work we shall not only consider fixed points and their locations in the plane, but we shall also consider points in motion, or the paths that are traced by such points as they move about in accordance with some definite law.

Exercises. Review of Graphs

1. Draw a graph to represent the functional relationship between x and y in the equation $y = 3x + 5$.

2. Draw a graph of the equation $x = 3y + 5$ on the axes used in Ex. 1. What relationship do you observe between the graphs? How do the distances from O cut off on the x axis and y axis compare in each case?

3. Using one pair of axes, draw the graphs of the equations $2x = 3y$, $2x + 1 = 3y$, and $2x - 1 = 3y$, and write a brief statement of the relationships between the graphs.

4. Using one pair of axes, draw the graphs of the equations $x + 3 = 4y$ and $2x - 4 = 3y$. Measure the coordinates of the point of intersection and compare them with the values of x and y found by solving the equations.

5. From Ex. 4 write a statement showing how a pair of linear equations can be solved by means of graphs.

6. Solve, if possible, the pair of equations $2x + 8 = 3y$ and $3x + 8 = 4.5y$. Draw the graphs of the equations on the same axes, and show how the graphs explain any difficulties encountered in the solution.

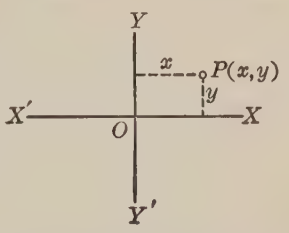
7. Draw a graph of the equation $x^2 + y^2 = 16$ by assigning to x the values $0, \pm\frac{1}{2}, \pm 1, \cdots, \pm 4$.

8. As in Ex. 7, draw a graph of the equation $x^2 + y^2 = 25$. What inference do you make regarding the geometric interpretation of an algebraic expression of the form $x^2 + y^2 = r^2$?

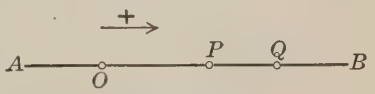
9. On the axes used in Ex. 8 draw the graph of the equation $3x - 4y = 0$. Find the coordinates of the two points of intersection of the graphs, and show that these points represent the solution of the two equations.

The above exercises are intended simply to review the technique of graphs and to show that between geometry and algebra there is a close relationship, which is the basis of analytic geometry.

Directed Lines. In considering the position of a point in our previous work, we made use of the convention that abscissas to the right of the y axis and ordinates above the x axis are positive, while abscissas to the left of the y axis and ordinates below the x axis are negative. In our work in this chapter we shall continue to use the same convention.



In order to make a clearer and more general statement about this idea of measurement in positive and negative directions along a line, let us upon any straight line, such as the line AB here shown, assume a point O as an origin from which distances are to be measured. Then, if we assign a positive direction, as from A to B, to the line, we say that AB is a *directed line.*



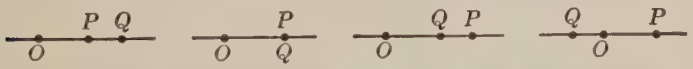
As usual, the word " line " used alone means a straight line; but it is often convenient to use " straight " for emphasis or clearness.

On the directed line AB, shown above, the length PQ is said to be *positive* if, in passing from P to Q, the passage is in the positive direction assigned to the line AB. If the passage is in the opposite direction, the length is said to be *negative.*

In the figures below, which show various positions of P and Q with respect to the origin O, the relation

$$PQ = OQ - OP$$

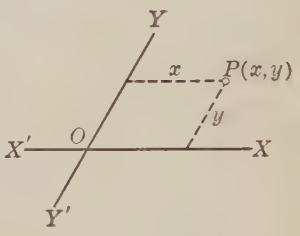
always defines the length PQ for any positions of P and Q.



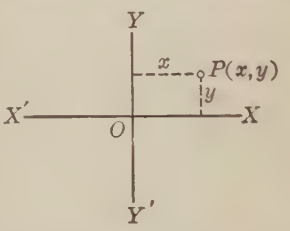
We see that in the first case the length PQ is positive, in the second it is zero, and in the third it is negative. Since both PQ and OQ in the fourth case are negative, we still have $PQ = OQ - OP$, and we see also that in all four cases $QP = -PQ$.

Cartesian Coordinates. We owe our analytic geometry chiefly to two mathematicians of the first half of the seventeenth century, René Descartes and Pierre de Fermat. The former was the author of a book entitled *La Géométrie*, which appeared in 1637. The Latin name of Descartes is *Cartesius*, and analytic geometry is sometimes called *Cartesian geometry*. The coordinates which we have used in locating points are also called *Cartesian coordinates*.

Oblique Coordinates. Although in our previous work the axes of reference were taken perpendicular to each other, this is not essential, and we may have axes oblique to each other, as in this figure. The abscissa and ordinate of the point P are then drawn parallel to their respective axes, and together they are known as the *oblique coordinates* of P.



Rectangular Coordinates. Ordinarily it is much simpler to use the perpendicular axes with which we are already familiar, and to locate points by coordinates measured from these axes. Such coordinates are called *rectangular coordinates*.



The term "coordinates" is applied not only to the line segments that locate the point P but also to the pair of numbers that represent these directed lengths. Thus, for every point P in the plane there is a pair of directed lengths, or a corresponding pair of real numbers which define its position; and, conversely, for any pair of real numbers, or the corresponding directed lengths which they represent, there is a point P in the plane.

Unless otherwise stated, we shall use only rectangular coordinates.

Exercises. Plotting Points

1. Locate the following points: $(2, 3)$, $(-3, 4)$, $(0, 1)$, $(\sqrt{2}, -5)$, $(-\sqrt{2}, -\sqrt{3})$.

2. What are the coordinates of a point on the x axis and 2 units to the right of the origin? on the x axis and 3 units to the left of the origin? on the y axis and 1 unit below the origin? on both the x axis and the y axis?

3. If a point moves in such a way that its ordinate is always zero, what is its path? If it moves in such a way that its abscissa is always zero, what is its path?

In Exs. 4–6 describe the path of a point which moves so that

4. Its ordinate is always -2; is always k.

5. Its abscissa is always 2; is always $-k$.

6. Its ordinate is always equal to its abscissa.

7. Describe the location of points for which the ordinates are always negative and the abscissas always positive.

8. Plot the four points $(3, 1)$, $(5, 3)$, $(3, 5)$, and $(1, 3)$ and connect them by lines in the order stated. What kind of quadrilateral is formed? Prove your statement.

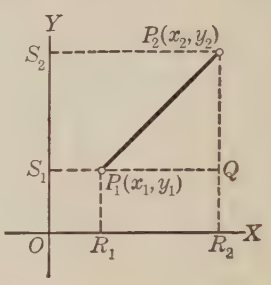
9. A square with the side s is to be drawn by plotting its vertices with reference to axes that intersect at the center of the square. If the sides are to be parallel to the axes, what coordinates should be taken for the vertices?

10. Solve Ex. 9 when the sides are to make angles of 45° with the axes.

11. If an equilateral triangle with the side s has one vertex at the origin, and one side on the x axis, what are the coordinates of the three vertices?

12. If a line segment AB drawn from the point $A(x, y)$ is bisected by the origin, what are the coordinates of B?

Projections of a Line Segment. In this figure, which shows a line segment P_1P_2 formed by connecting the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, the segment R_1R_2 cut off by the ordinates of P_1 and P_2 is called the *projection of P_1P_2 on the x axis.* Similarly, S_1S_2 is called the *projection of P_1P_2 on the y axis.*



The convenience of using perpendicular axes is here apparent.

Since R_1R_2 and S_1S_2 are directed segments, we see that

$$R_1R_2 = OR_2 - OR_1 = x_2 - x_1,$$

and $$S_1S_2 = OS_2 - OS_1 = y_2 - y_1;$$

that is, expressed in words, these relations show that

The projection of a line segment upon the x axis is the difference between the abscissas of its ends.

The projection of a line segment upon the y axis is the difference between the ordinates of its ends.

Length of a Line Segment. If in the above figure we let l represent the length of P_1P_2, we have

$$l^2 = \overline{P_1Q}^2 + \overline{QP_2}^2.$$

Then, since $P_1Q = R_1R_2 = x_2 - x_1$

and $QP_2 = S_1S_2 = y_2 - y_1,$

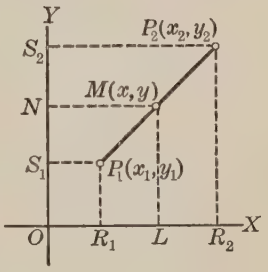
we have $l^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$

Hence $$l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

which is a formula for the length of the line segment in terms of the abscissas and ordinates of its ends.

This formula is valid for any position of P_1 and P_2. It is advisable for the student to verify it for other positions of the points, as in Ex. 2 on page 169, and thus to realize the generality of such a formula.

Midpoint of a Line Segment. If we let $M(x, y)$ be the midpoint of the line segment P_1P_2, which was considered on page 168, and draw the coordinates of the point M, as shown in this figure, we see at once that



$$R_1L = x - x_1,$$

$$LR_2 = x_2 - x,$$

$$S_1N = y - y_1,$$

and $$NS_2 = y_2 - y.$$

Furthermore, since the ordinates are parallel and the abscissas are parallel, and since $P_1M = MP_2$, we have

$$R_1L = LR_2,$$

or $$x - x_1 = x_2 - x,$$

and similarly, $$S_1N = NS_2,$$

or $$y - y_1 = y_2 - y.$$

Solving these two equations, we then have the following formulas for the coordinates x and y of the midpoint:

$$x = \tfrac{1}{2}(x_1 + x_2),$$

and $$y = \tfrac{1}{2}(y_1 + y_2).$$

Exercises. Line Segments

1. From the above formulas for x and y write a rule for finding the coordinates of the midpoint of a line segment.

2. Draw four figures to illustrate different cases of the formula for l given on page 168, taking P_1 in the other three quadrants or at O, and taking P_2 in any of the quadrants or on either axis. In each case show that the formula is valid.

3. Find the lengths of the projections on the two axes of the line joining the point $(-3, 2)$ to the point $(1, 3)$, and also find the length of the segment.

Find the lengths of the line segments determined by the following pairs of points:

4. $(-1, 1), (3, -5)$.
5. $(5, -3), (-2, 3)$.
6. $(-3, -5), (3, -2)$.
7. $(\frac{1}{3}, -\frac{3}{4}), (-\frac{3}{2}, \frac{4}{3})$.
8. $(\frac{1}{2}, \frac{1}{2}\sqrt{2}), (2, -\frac{1}{2}\sqrt{2})$.
9. $(2\frac{1}{2}, \frac{1}{2}\sqrt{3}), (-3\frac{1}{2}, -\frac{1}{2}\sqrt{3})$.
10. $(2+\sqrt{3}, 1), (2-\sqrt{3}, 2)$.
11. $(3+\sqrt{5}, 1), (3-\sqrt{5}, -6)$.

12. Find the coordinates of the midpoints of the sides of the triangle formed by joining the points $(2, 3)$, $(4, -5)$, and $(-3, -6)$.

13. Find the length of each of the medians of the triangle mentioned in Ex. 12.

14. If one end of a line segment is the point $(5, 7)$, and the midpoint is the point $(6, 4)$, what are the coordinates of the other end of the line?

15. Find the lengths of the sides of the triangle determined by the three points $(1, 2)$, $(-2, 3)$, and $(-5, -6)$.

16. Prove that the triangle which is determined by the points $(1, 1)$, $(-2, 1)$, and $(1, -2)$ is isosceles.

17. If the three vertices of a triangle are $A(1, 1)$, $B(4, 4)$, and $C(m, 5)$, for what value or values of m will the triangle be isosceles when AB is the base? when AC is the base? when BC is the base?

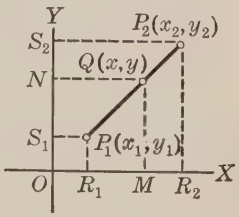
18. Prove that the triangle with the vertices $(1, 1)$, $(2, 2)$, and $(\frac{3}{2} - \sqrt{15}, \frac{3}{2} + \sqrt{15})$ is isosceles.

19. If the vertices of a triangle are $(2, 3)$, $(4, 3)$, and (a, b), for what value or values of a and b will it be equilateral?

20. In the isosceles triangle which has the vertices $(0, 0)$, $(2, 0)$, and $(1, 4)$, what is the length of the line segment that joins the midpoints of the two equal sides?

21. What is the length of the diagonal of a square whose vertices are $(0, 0)$, $(-8, 0)$, $(-8, 8)$, and $(0, 8)$?

Dividing a Line in a Given Ratio. To divide a line segment in a given ratio, we may locate the point of division by finding its coordinates. For example, in this figure, if we let $Q(x, y)$ be the point that divides the line segment determined by the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the ratio $\frac{m}{n}$, we see that



$$\frac{P_1Q}{QP_2} = \frac{R_1M}{MR_2} = \frac{x - x_1}{x_2 - x},$$

and

$$\frac{P_1Q}{QP_2} = \frac{S_1N}{NS_2} = \frac{y - y_1}{y_2 - y}.$$

Now $\frac{P_1Q}{QP_2}$ is the same as the given ratio $\frac{m}{n}$, and hence

$$\frac{x - x_1}{x_2 - x} = \frac{m}{n} \quad \text{and} \quad \frac{y - y_1}{y_2 - y} = \frac{m}{n}.$$

Solving these two equations for x and y, the coordinates of the point of division, we then have the formulas

$$x = \frac{nx_1 + mx_2}{m + n},$$

and

$$y = \frac{ny_1 + my_2}{m + n}.$$

It should also be observed that these formulas hold when the given ratio is either positive or negative. If it is positive, the point of division is internal; if it is negative, the point is external (that is, it lies on the given line segment produced).

For example, find the point which divides the line joining the points (3, 5) and (− 5, 8) in the ratio $\frac{3}{4}$.

In the above formulas, we have $x_1 = 3$, $x_2 = -5$, $y_1 = 5$, $y_2 = 8$, $m = 3$, and $n = 4$. We then have

$$x = \frac{4 \cdot 3 + 3(-5)}{3 + 4} = \frac{12 - 15}{7} = \frac{-3}{7},$$

and

$$y = \frac{4 \cdot 5 + 3 \cdot 8}{3 + 4} = \frac{20 + 24}{7} = \frac{44}{7}.$$

The required point is therefore $(-\frac{3}{7}, 6\frac{2}{7})$.

Exercises. Dividing a Line Segment

For each of the following pairs of points find the coordinates of the point which divides the line segment joining the points in the ratio specified, and draw the figure:

1. $(3, 2), (6, 8); \frac{2}{3}$.
2. $(-2, 5), (\frac{1}{2}, \frac{2}{3}); \frac{3}{1}$.
3. $(\frac{2}{3}, \frac{3}{4}), (\frac{1}{2}, -\frac{1}{4}); \frac{4}{1}$.
4. $(0, 0), (a, b); \frac{1}{r}$.
5. $(-3, \sqrt{2}), (\sqrt{2}, 3); -\frac{1}{2}$.
6. $(2.4, 3.2), (3.5, -2.8); \frac{3}{2}$.
7. $(10, -2), (-12, -15); \frac{6}{7}$.
8. $(\sqrt{5}, \sqrt{2}), (\sqrt{2}, \frac{4}{5}\sqrt{5}); \frac{\sqrt{2}}{\sqrt{5}}$.

9. Find the coordinates of the two points which trisect the segment joining the points (3, 5) and (8, − 7).

10. When $m = n$, show that the formulas on page 171 reduce to those on page 169.

11. Find the midpoints of the sides of the $\triangle ABC$ formed by joining the points $A(4, 8)$, $B(6, 10)$, and $C(5, -4)$.

12. In Ex. 11 find the length of the medians, and also find the coordinates of the trisection points of the medians. Show that the three medians of the triangle meet in a point.

13. Find the coordinates of the midpoints of the sides of the quadrilateral formed by joining in order the points (2, 2), (− 3, 1), (− 3, − 4), and (2, − 5). Show that the quadrilateral which is determined by these four midpoints is a parallelogram.

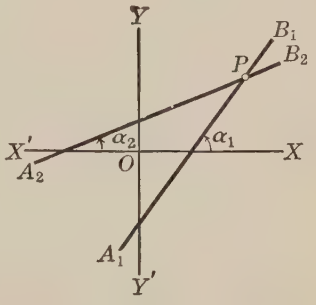
14. If one end of a line segment is the point (3, 8), and the midpoint is (4, 5), what are the coordinates of the other end of the line?

15. Show that the points (− 2, − 2) and (8, 2) are equidistant from the point (1, 5).

16. Show that the length of the line segment joining the midpoints of the sides AB and BC of the $\triangle ABC$ having the vertices $A(-3, -1)$, $B(3, -3)$, and $C(4, 6)$ is equal to $\frac{1}{2}AC$.

2. Inclination and Slope

Inclination of a Line. The angle between the upward direction of a line and the positive direction of the x axis is called the *inclination* of the line.



Thus, in this figure the inclination of the line A_1B_1 is α_1.

If on the line A_1B_1 we now take a point P and let A_1B_1 rotate about P in a clockwise direction until it reaches the position A_2B_2, as shown in the figure, the inclination of the line decreases.

That is, the angle made by the line with the x axis is decreasing, and hence we have $\alpha_2 < \alpha_1$.

If, on the other hand, a line is allowed to rotate about a point on the line in a counterclockwise direction, it is evident that the inclination increases.

For example, if the line A_2B_2 rotates counterclockwise about P to the position A_1B_1, we have $\alpha_1 > \alpha_2$.

If the rotation of a line is continued in a clockwise direction until the line is parallel to the x axis, the inclination of the line becomes zero.

In the same way, if a counterclockwise rotation of a line is continued until the line is parallel to the y axis, the inclination becomes 90°. If the rotation is continued in the same direction until the line is parallel to the x axis, the inclination becomes 180°; that is, the inclination α of a line may vary from 0° to 180°.

Although we might define " inclination " in such a way that it would include a negative inclination of a line, this is neither necessary nor desirable. It is evident that if a line cuts the x axis, we can always express the angle α as positive; and if the line is parallel to the x axis, we may call the angle a zero angle.

Slope of a Line. Instead of making direct use of the inclination of a line, we shall find that it is often more convenient to use the tangent of the inclination, which is called the *slope* of the line. That is, following the usual custom of representing the slope of a line by m, we have

$$m = \tan \alpha.$$

The slope of a line is therefore positive or negative according as α is acute or obtuse (page 88).

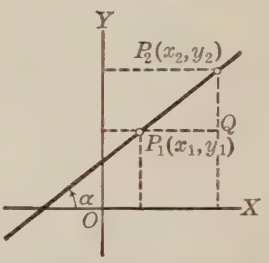
If in the figure below we let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points determining a line with the inclination α, we have

$$m = \tan \alpha = \tan QP_1P_2 = \frac{QP_2}{P_1Q}.$$

Also, $QP_2 = y_2 - y_1$

and $P_1Q = x_2 - x_1$;

whence $$m = \frac{y_2 - y_1}{x_2 - x_1}.$$



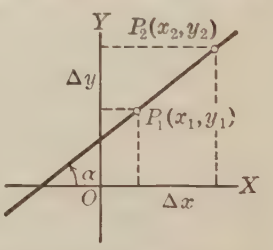
In other words, we see that

The slope of a line determined by two given points is the quotient of the difference between the ordinates of the points divided by the difference between the abscissas.

Difference Quotient. In the last formula for m, above, the difference $y_2 - y_1$ between the y's is often represented by Δy (read "delta y"), and the difference $x_2 - x_1$ between the x's by Δx. When this is done, the formula for m becomes

$$m = \frac{\Delta y}{\Delta x}.$$

In this connection the expression $\frac{\Delta y}{\Delta x}$ is called the *difference quotient.*



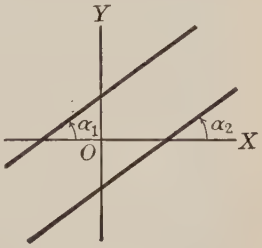
The difference quotient will be considered further in relation to the calculus in Chapter VI.

Parallel Lines. If two lines are parallel, their inclinations, as shown in this figure, are corresponding angles, and hence

$$\alpha_1 = \alpha_2.$$

Then $\tan \alpha_1 = \tan \alpha_2,$

or $m_1 = m_2.$



That is, if two lines are parallel, their slopes are equal.

Conversely, if the slopes of two lines are equal, the lines are parallel.

For example, let us take the line determined by $A(2, 8)$ and $B(1, 6)$, and the line determined by $C(0, 2.5)$ and $D(4, 10.5)$. We then have for AB,

$$m_1 = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 8}{1 - 2} = \frac{-2}{-1} = 2,$$

and for CD, $$m_2 = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10.5 - 2.5}{4 - 0} = \frac{8}{4} = 2.$$

That is, since $m_1 = m_2$, we see that the line AB is parallel to the line CD.

Perpendicular Lines. If two lines are perpendicular to each other, as shown in this figure, it is evident from geometry that

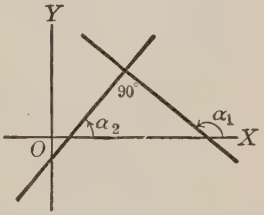
$$\alpha_1 = 90° + \alpha_2;$$

whence $\tan \alpha_1 = \tan(90° + \alpha_2)$

$$= -\cot \alpha_2.$$

Since $\cot \alpha_2 = \dfrac{1}{\tan \alpha_2},$

we have $$m_1 = -\frac{1}{m_2}.$$



That is, if two lines are perpendicular to each other, the slope of one line is the negative reciprocal of the slope of the other.

Conversely, if the slope of one line is the negative reciprocal of the slope of another, the two lines are perpendicular to each other.

For example, we see at once that a line with the slope $-\frac{1}{2}$ is perpendicular to either of the lines AB and CD mentioned above.

Exercises. Inclination and Slope

Find the slope of a line which has an inclination of

1. 30°. 2. 60°. 3. 135°. 4. 150°. 5. 68° 30′. 6. $105\frac{1}{4}°$.

Find the inclination of a line which has the slope

7. $\frac{1}{2}$. 8. $\frac{2}{3}$. 9. $-\frac{1}{3}$. 10. -2. 11. $-\frac{5}{2}$. 12. 0.

13. Find the slope of the line determined by the points (1, 2) and (− 3, 4).

14. Find the slope and the inclination of the line determined by the point (5, 5) and the origin.

15. Find the slopes of the sides of the triangle determined by the points (3, 2), (− 3, 3), and (− 1, − 4).

16. Find the slopes of the perpendicular bisectors of the sides of the triangle in Ex. 15.

17. Find the slopes of the medians of the triangle in Ex. 15.

18. Show that the quadrilateral formed by joining the midpoints of the sides of the quadrilateral which has its vertices at the points (1, 1), (2, 3), (6, 1), and (1, − 2) is a parallelogram.

19. Show that the line determined by the points (3, 2) and (− 1, 4) is perpendicular to the line determined by the points (− 1, 1) and (2, 7).

20. For what value of a will the line joining (2, a) to (1, 3) be parallel to the line joining (− 1, a) to (3, 1)?

21. Show that the line through (2, 4) and (5, 3) is perpendicular to the line through (4, − 2) and (3, − 5).

Find $\frac{\Delta y}{\Delta x}$ *for the line determined by the points*

22. (1, 2) and (3, − 4).
23. (2, a) and (3, b).
24. (− 3, 5) and (6, $-\frac{1}{2}$).
25. (− 2, 3) and (− 3, − 5).
26. (2, $-\frac{2}{5}$) and ($1\frac{1}{2}$, $-8\frac{1}{4}$).
27. $(\sqrt{2}+1, 2)$ and $(\sqrt{2}-1, 4)$.

3. Straight-Line Loci

Locus. The curve which contains all points in the plane that satisfy a given condition, and which contains no points that do not satisfy the given condition, is often called the *locus* (the plural is *loci*, usually given the English pronunciation, lō′-sī) of the points.

For example, the graph of the formula $A = \pi r^2$ on page 28 may be called the locus of the formula, since it contains all points (or would contain them if extended far enough) which satisfy the condition that $A = \pi r^2$, and it contains no points which do not satisfy this condition.

Straight-Line Loci. In analytic geometry it is convenient to regard a straight line as a special case of a curve. A locus may therefore be a straight line, — in fact, the loci of elementary plane geometry are frequently straight lines.

For example, as the student already knows from plane geometry, the locus of a point that is equidistant from two given points is easily shown to be the perpendicular bisector of the line segment joining the points.

The idea of locus is also extended to solid geometry. In this case the locus just described becomes the plane that is perpendicular to the line segment joining the two given points and that passes through the midpoint of this segment.

Instead of a single line there may be two or more lines that form the locus under certain conditions. For example, in plane geometry it is proved that the locus of a point which is equidistant from two intersecting lines is made up of the two lines that bisect both pairs of vertical angles formed by the given lines.

We may also have a curve for a locus in plane geometry. For example, in a plane the locus of a point at a given distance from a fixed point is the circle of which the fixed point is the center and the given distance is the radius.

The corresponding locus in solid geometry is the sphere (a surface) with the fixed point as center and the given distance as radius.

Finding the Locus of an Equation. Our work with graphs has already furnished one example of finding the locus of an equation. In such work we are given a formula, or an equation, and from it we are required to determine the corresponding locus (the graph).

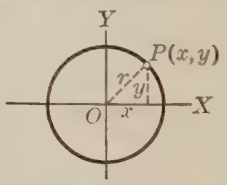
In constructing loci it is sometimes possible to perform the construction mechanically, as in drawing a straight line along a ruler after two points of the line have been located, or in drawing a circle with the compasses after the center and radius have been determined.

If a mechanical construction is impossible, we may plot a sufficient number of points, as in the graph on page 28, and then sketch the approximate locus by connecting these points by a smooth curve.

Finding the Equation of a Locus. We shall frequently need to consider cases in which we are given a locus, or more often we are given a problem stating the conditions that must be satisfied, and are required to find the corresponding equation. If we can formulate in mathematical language some statement about the coordinates x and y of a point P which moves so as to satisfy the given conditions, the resulting equation in x and y will be the required equation of the locus.

For example, if a point $P(x, y)$ must move so as to be always at the distance r from the point $(0, 0)$, we see from this figure that for any position of P

$$x^2 + y^2 = r^2.$$



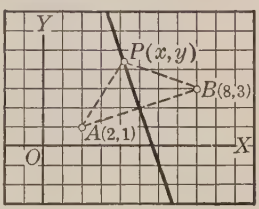
We thus find that the equation of the given locus, which is the circle with the center O and the radius r, is $x^2 + y^2 = r^2$.

It is with these two problems, and particularly the second one (finding the equation of a locus), that analytic geometry is chiefly concerned.

We shall now consider two special cases of a problem that will illustrate the procedure.

Discussion of a Special Problem. Following the suggestions given on page 178, in finding the equation of the locus of a point which moves so as always to be equidistant from the points $A(2, 1)$ and $B(8, 3)$, we may first assume that $P(x, y)$ is any point of the locus, and hence

$$PA = PB.$$



Then, as was shown on page 168,

$$PA = \sqrt{(2-x)^2 + (1-y)^2},$$

and $PB = \sqrt{(8-x)^2 + (3-y)^2}$;

whence $\sqrt{(2-x)^2 + (1-y)^2} = \sqrt{(8-x)^2 + (3-y)^2}$.

By squaring and simplifying this equation we then have

$$3x + y - 17 = 0,$$

which is the required equation of the given locus.

Since P was taken as any point on the locus, the coordinates of every point on the locus satisfy the equation.

That is, if the point (4, 5) is on the locus, the equation is satisfied by $x = 4$ and $y = 5$, which can be shown to be the case.

Conversely, if the coordinates of any point $P_1(x_1, y_1)$ satisfy the equation, that is, if $3x_1 + y_1 - 17 = 0$ is a true equation, it follows that the equation

$$\sqrt{(2-x)^2 + (1-y)^2} = \sqrt{(8-x)^2 + (3-y)^2}$$

is also true when x_1 and y_1 are substituted for x and y; that is, P_1 is a point on the locus.

For example, since $x = 5$ and $y = 2$ can be shown to satisfy the equation, the point (5, 2) is on the locus.

Since we know from plane geometry that the locus is a straight line (page 177), we may now construct the locus with a ruler through the points (4, 5) and (5, 2).

The student should notice that we may say that a point lies on a locus when and only when its coordinates satisfy the equation of the locus.

Further Discussion of the Problem. In solving the problem given on page 179, we might have observed at first that the locus is the perpendicular bisector of the line segment which joins the given points $A(2, 1)$ and $B(8, 3)$. That is, the locus must pass through the midpoint of AB.

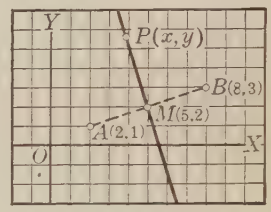
Letting M represent this midpoint, we see from page 169 that for the abscissa x we have

$$x = \tfrac{1}{2}(x_1 + x_2) = \tfrac{1}{2}(2 + 8) = 5,$$

and for the ordinate y,

$$y = \tfrac{1}{2}(y_1 + y_2) = \tfrac{1}{2}(1 + 3) = 2;$$

that is, one point of the locus is the point $M(5, 2)$.



Since the locus is perpendicular to AB, the slope of the locus must, as shown on page 175, be the negative reciprocal of the slope of AB. In finding the slope m of AB, we have

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{8 - 2} = \frac{2}{6} = \frac{1}{3},$$

and hence we see that the slope m_1 of the locus must be -3.

Taking $P(x, y)$ as any point except M on the locus, we may then express the slope of the locus in terms of the coordinates of M and P; that is, since

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1},$$

we have

$$-3 = \frac{y - 2}{x - 5};$$

whence

$$3x + y - 17 = 0,$$

which is the required equation, and is, of course, the same as that found on page 179.

As on page 179, we may now construct the locus by taking a second point whose coordinates satisfy the equation.

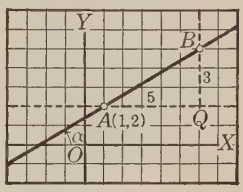
Pages 179–181 are intended to serve only as an introduction to analytic methods. The procedure in general cases will be considered later in detail.

Determining a Straight-Line Locus. There are various conditions that determine a straight-line locus, but they all depend upon the two following propositions:

1. *A straight line is determined by two points.*

2. *A straight line is determined by a point and the slope of the line.*

In order to illustrate the procedure in constructing a locus by the second of these methods, without first finding the equation of the line as was done on page 180, we shall now consider another special problem. For example, suppose that we are required to construct the line that passes through the point $A(1, 2)$ and has the slope 0.6.



Since $m = \tan \alpha = 0.6 = \frac{6}{10} = \frac{3}{5}$, the problem reduces to constructing through A a line meeting the x axis in such a way that the tangent of the inclination is $\frac{3}{5}$.

Drawing a line through A parallel to the x axis, we first take a point Q on this line such that $AQ = 5$. Drawing a line through Q parallel to the y axis, we then take a point B on this line such that $QB = 3$. The required locus may then be drawn through the two points A and B, as shown in the figure above.

We must now prove that AB is the required locus. From the figure we see that $\Delta y = 3$ and $\Delta x = 5$, and hence

$$m = \frac{\Delta y}{\Delta x} = \frac{3}{5} = 0.6;$$

that is, AB passes through $A(1, 2)$ and has the slope 0.6, thus meeting the conditions stated in the problem.

We now see that in the problem on page 180, where we found that $M(5, 2)$ was one point of the locus and that $m_1 = -3 = -\frac{3}{1}$, we might have constructed the locus by taking a point 1 unit to the left of (5, 2) and 3 units above it, and then drawing a straight line through (5, 2) and the point (4, 5) thus found.

Exercises. Straight-Line Loci

Find the equation of the locus of a point that is always

1. 3 units above the x axis.

2. 5 units below the x axis.

3. 1 unit to the left of the y axis.

4. On the x axis.

5. On the y axis.

6. On both axes.

In each of the following cases construct the line that passes through the given point with the slope or inclination specified:

7. $(1, 2)$, $m = \frac{2}{3}$.

8. $(-2, 3)$, $m = -\frac{1}{2}$.

9. $(5, -6)$, $m = 3$.

10. $(-2, -3)$, $m = -4$.

11. $(2, 3)$, $\alpha = 45°$.

12. $(0, 2)$, $\alpha = 120°$.

13. $(2, 0)$, $\alpha = 135°$.

14. $(1, -5)$, $\alpha = 60°$.

15. Find the equation of the locus of a point that moves so as always to be equidistant from the points $(2, 1)$ and $(-3, -4)$; equidistant from the x and y axes.

16. Find the equation of the line through the point $(2, 3)$ and parallel to the line joining $(1, -1)$ to $(-4, -2)$.

In each of the following cases find the equation of the perpendicular bisector of the line that joins the two points:

17. $(1, 2)$, $(4, 5)$.

18. $(2, 3)$, $(3, -6)$.

19. $(-2, 1)$, $(5, 4)$.

20. $(0, 2)$, $(2, 0)$.

21. $(2, 2)$, $(-2, -2)$.

22. (a, b), (c, d).

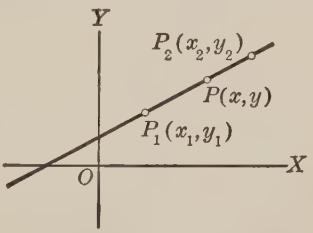
23. Find the equation of the line that passes through the origin and is perpendicular to the line through the points $(2, 3)$ and $(-3, 4)$.

24. Find the equation of the line determined by the points $(1, 2)$ and $(3, 4)$.

Assume that $P(x, y)$ is on the line, and express the condition that the slope of the line determined by $(1, 2)$ and (x, y) is equal to the slope of the line determined by $(1, 2)$ and $(3, 4)$.

The Two-Point Formula. We shall now consider the development of several standard formulas by means of which we shall be able to write the equations of the straight lines that are determined by certain given conditions. The simplest of these, is the formula that enables us to write the equation of the line that passes through two given points.

If we take $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ as the two given points and assume that $P(x, y)$ is any other point of the line, we may express the slope m of the line in terms of the coordinates of P_1 and P_2. That is, we have



$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Since any two points of a straight line determine the line, we may also express the slope in terms of the coordinates of P and P_1; that is, we have

$$m = \frac{y - y_1}{x - x_1}.$$

Equating these two expressions for m, we then have

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Since this formula involves the variables x and y and the known coordinates (x_1, y_1) and (x_2, y_2) of the two given points P_1 and P_2, it is known as the *two-point formula.*

In finding the equation of the straight line which passes through the points (1, 2) and (6, 5), we have

$$\frac{y - 2}{x - 1} = \frac{5 - 2}{6 - 1} = \frac{3}{5};$$

whence $$3x - 5y + 7 = 0,$$

which is the required equation. It is easily shown that this equation is satisfied both by $x = 1$, $y = 2$ and by $x = 6$, $y = 5$.

In the figure on page 181, it is seen that AB passes through the points (1, 2) and (6, 5), and hence $3x - 5y + 7 = 0$ is the equation of that line.

The Intercept Formula. If one of the two points that determine a straight line lies on the x axis, and if the other point lies on the y axis, the two-point formula for the straight line can be somewhat simplified.

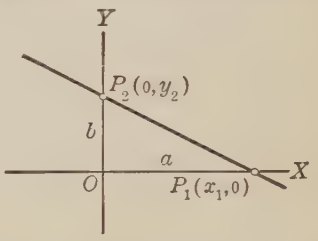
That is, if the point P_1 is on the x axis, we have the point $P_1(x_1, 0)$; if P_2 is on the y axis, we have the point $P_2(0, y_2)$. The two-point formula given on page 183 therefore becomes

$$\frac{y-0}{x-x_1}=\frac{y_2-0}{0-x_1},$$

or
$$\frac{y}{x-x_1}=-\frac{y_2}{x_1};$$

whence
$$xy_2+x_1y=x_1y_2,$$

or
$$\frac{x}{x_1}+\frac{y}{y_2}=1.$$



From the figure we see that the line cuts off the lengths OP_1 and OP_2 on the x axis and y axis respectively. It is customary to designate these lengths, which are called the *x intercept* and the *y intercept* of the line, by the letters a and b. That is, in the last equation above, if we let

$$x_1 = a$$

and
$$y_2 = b,$$

we have
$$\frac{x}{a}+\frac{y}{b}=1,$$

which involves the variables x and y and the known intercepts a and b. This formula for the straight line is therefore called the *intercept formula.*

If we wish to find the equation of a line which cuts off the intercept 5 on the x axis and the intercept 7 on the y axis, we have

$$\frac{x}{5}+\frac{y}{7}=1,$$

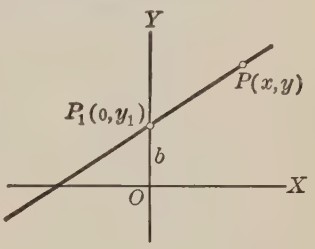
or
$$7x+5y=35,$$

which is the required equation. The locus may be easily constructed by taking $a = 5$ and $b = 7$ and drawing the line accordingly.

The Intercept-Slope Formula. We shall now develop a formula by means of which we can write the equation of a line when the y intercept and the slope are given.

If we let $P_1(0, y_1)$ be the point where the line cuts the y axis and $P(x, y)$ any other point on the line, the slope m of the line in terms of the coordinates of P and P_1 then becomes

$$m = \frac{y - y_1}{x - 0}.$$



Now $OP_1 = y_1 = b,$

and hence $m = \frac{y - b}{x}.$

From this expression for m we may now derive the following formula, which is called the *intercept-slope formula:*

$$y = mx + b.$$

For example, the equation of the line which has the slope $-\frac{1}{2}$ and the y intercept 2, is $y = -\frac{1}{2}x + 2$, or $x + 2y - 4 = 0$.

Degree of the Equation of a Straight Line. Since in a plane every straight line that is not parallel to the y axis has an intercept b on this axis and a slope m, the equation

$$y = mx + b$$

represents any straight line that is not parallel to the y axis.

Similarly, every straight line parallel to the y axis has an x intercept a, and may be represented by the equation $x = a$.

Every line in the plane, therefore, may be expressed by one of these two equations, from which we see that

The equation of any straight line in the plane is of the first degree in x and y.

This theorem shows why an equation of the first degree in x and y is called a *linear equation*, a term which is also extended to include any equation of the first degree. The converse of this theorem will be considered on pages 189 and 190.

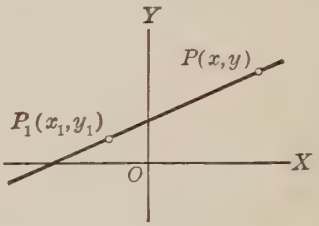
The Point-Slope Formula. We may now consider the general case of the problem considered on page 180; namely, finding the equation of a line which passes through a given point and has a given slope.

Taking $P_1(x_1, y_1)$ as the given point and letting $P(x, y)$ be any other point on the line, the slope m of the line in terms of the coordinates of the points P_1 and P is

$$m = \frac{y - y_1}{x - x_1},$$

from which we can easily derive the *point-slope formula*

$$y - y_1 = m(x - x_1).$$



Thus, the equation of the line which passes through the point (5, 2) and has the slope -3 is

$$y - 2 = -3(x - 5),$$

or

$$3x + y - 17 = 0,$$

which is the same as the equation found on page 180. The advantages of having a general formula that can be applied to the solution of all cases of this type are apparent.

Choice of Formulas. In writing the equation of a straight line we may use the formula which is most convenient, the choice depending, of course, upon the given conditions. That is, if we are given two points, we should naturally use the two-point formula, and similarly in the other cases. The following summary of the formulas thus far developed will be convenient for reference:

Given	Formula to be Used
1. Two points;	$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$
2. The x and y intercepts;	$\frac{x}{a} + \frac{y}{b} = 1.$
3. The y intercept and the slope;	$y = mx + b.$
4. One point and the slope;	$y - y_1 = m(x - x_1).$

Exercises. Formulas for a Straight Line

In each of the following cases find the equation of the line determined by the points:

1. $(1, 3), (-3, 4)$.
2. $(-5, 6), (7, 8)$.
3. $(2, 0), (-6, 8)$.
4. $(6, -8), (0, 2)$.
5. $(\frac{4}{5}, \frac{2}{3}), (-\frac{3}{2}, \frac{5}{4})$.
6. $(8, 8), (-\frac{1}{2}, -\frac{3}{4})$.
7. $(-4, 0.6), (5.2, -1.8)$.
8. $(1+\sqrt{2}, 3), (\sqrt{2}-1, 5)$.

9. Find the equations of the sides of the triangle which has its vertices at the points $(3, -2)$, $(4, 5)$, and $(-3, -1)$.

10. In Ex. 9 find the equations of each of the three medians of the triangle.

In each of the following cases find the equation of the line determined by the specified intercept and slope:

11. $b = -3$, $m = 2$.
12. $b = 5$, $m = -\frac{2}{3}$.
13. $b = 6$, $m = -4$.
14. $b = 0$, $m = -1$.
15. $b = \frac{1}{2}$, $m = \frac{1}{3}$.
16. $b = -\frac{2}{3}$, $m = \frac{5}{6}$.
17. $b = -2.8$, $m = -2$.
18. $b = 4.6$, $m = 0.8$.

In each of the following cases find the equation of the line determined by the x and y intercepts:

19. $a = 2$, $b = 3$.
20. $a = 3$, $b = -5$.
21. $a = -4$, $b = 6$.
22. $a = -1$, $b = -3$.
23. $a = \frac{4}{5}$, $b = \frac{1}{2}$.
24. $a = 1.8$, $b = -5.4$.

In each of the following cases find the equation of the line which passes through the given point and has the slope specified:

25. $P_1(2, 3)$, $m = \frac{2}{3}$.
26. $P_1(2, -3)$, $m = \frac{2}{3}$.
27. $P_1(-3, 2)$, $m = \frac{3}{2}$.
28. $P_1(0, 4)$, $m = 2$.
29. $P_1(0, 0)$, $m = -2$.
30. $P_1(8, 0)$, $m = -1.2$.

31. Find the equations of the perpendicular bisectors of the sides of the triangle which has its vertices at the points $(-3, 4)$, $(-1, -5)$, and $(6, -1)$.

32. In Ex. 31 find the equations of the lines drawn through the vertices of the triangle parallel to the opposite sides.

33. Find the equations of the lines which join the mid-points of the consecutive sides of the quadrilateral which has the vertices $(3, 4)$, $(5, -6)$, $(-2, -3)$, and $(-4, 2)$.

34. A right triangle in the first quadrant has the vertex of its smallest angle at the origin and the vertex of the right angle at the point $(4, 0)$. If the length of the hypotenuse is 5 units, what is the equation of the perpendicular drawn from the vertex of the right angle to the hypotenuse?

35. In the triangle which has its vertices at the points $(0, 0)$, $(5, 0)$, and $(4, 3)$ find the equations of the perpendiculars from the vertices to the opposite sides. Show that all these equations are satisfied by the same pair of values of x and y, and state the geometric fact which this proves.

36. Solve Ex. 35 for the perpendicular bisectors of the sides of the triangle and also for the medians.

37. Solve Ex. 35 for the general triangle which has its vertices at the points $(0, 0)$, $(x_1, 0)$, and (x_2, y_2), and thus prove that in any triangle the perpendiculars from the vertices to the opposite sides meet at one point.

Any triangle can, of course, be placed so that one vertex is at the origin and one side is on the x axis.

38. Find the equations of the perpendicular bisectors of the sides of the triangle in Ex. 37 and show that these lines meet at one point.

39. Find the equations of the medians of the triangle in Ex. 37 and show that these lines meet at a point which is a trisection point of each median.

Locus of an Equation of the First Degree. On page 185 we found that the equation of any straight line in the plane is of the first degree in x and y. We have not yet proved the converse of this theorem; namely,

The locus of every equation of the first degree in x and y is a straight line.

That such a statement needs to be proved is seen from the fact that the equation of a circle is of the second degree, but, as we shall presently find, the locus of an equation of the second degree is not necessarily a circle.

In the special case that was considered on pages 179 and 180, we constructed the locus of an equation of the first degree. In this case, however, we knew from plane geometry that the locus was a straight line. If we had not known this, we might have proceeded to plot a number of points, and thus found that the locus was probably a straight line, but such a method would be laborious and it would not prove that the locus actually was a straight line. If we can prove that the locus of every equation of the first degree in x and y is a straight line, we may then make a mechanical construction with the ruler after we have determined two points or the two intercepts.

In proving the above theorem, we shall take the equation

$$Ax + By + C = 0,$$

where A, B, and C are arbitrary constants, as the general form of an equation of the first degree in x and y. Every such equation may be reduced to this form by collecting the coefficients of x and y and then transposing all terms to the left-hand member of the equation.

The following examples show how several different types of linear equations in x and y may be reduced to the general form:

$2x - 3y = 6$, $\qquad$ $2x - 3y - 6 = 0$;

$\frac{1}{2}x + 5 = y + 7$, $\qquad$ $x - 2y - 4 = 0$;

$\frac{3}{4}x + 6 - y = 3y + x - 4$, $\qquad$ $x + 16y - 40 = 0$.

Proof of the Converse. In order to prove the converse stated on page 189, we need only compare the equation

$$Ax + By + C = 0,$$

the locus of which we are to prove is a straight line, with the two equations

$$y = mx + b$$

and

$$x = a,$$

which, as we found on page 185, represent all possible straight lines in the plane.

In the general equation it should first be observed that for any value of x there is always a corresponding value of y, and hence there is always a locus that corresponds to any equation of the first degree in x and y.

If this fact were not established, we might find ourselves talking about a locus that did not exist.

One of the various possibilities in the general equation

$$Ax + By + C = 0$$

is that $B = 0$. Solving for x in this case, we then have

$$x = -\frac{C}{A}.$$

This equation is of the form $x = a$, which was shown (page 185) to be the equation of a line parallel to the y axis.

Similarly, if $B \neq 0$, we may solve for y and have

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

This equation is of the form $y = mx + b$, which was shown (page 185) to be the equation of the line with the slope m and the y intercept b.

Hence in both cases, which cover all possible ones that can arise, the locus of $Ax + By + C = 0$ is a straight line.

We need not consider the case of $A = 0$, since this gives the equation of a line parallel to the x axis, which is included in the case $B \neq 0$.

Formula for the Slope. When the general equation of the first degree in x and y is written in the form

$$y = -\frac{A}{B}x - \frac{C}{B},$$

we know from page 190 that the equation is of the form

$$y = mx + b.$$

That is, in the general equation, we have

$$m = -\frac{A}{B}.$$

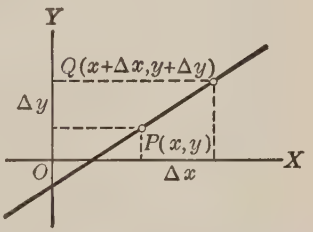
In other words, this formula for m shows that

The slope of the line represented by $Ax + By + C = 0$ is minus the coefficient of x divided by the coefficient of y.

Thus, given an equation like $2x - 3y - 7 = 0$, we can determine its slope, which in this case is $\frac{2}{3}$, by inspection.

As an alternative proof, let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$, where Δx and Δy are the differences between the abscissas and the ordinates respectively, be any two points on the straight line which may be represented by the equation

$$Ax + By + C = 0.$$



Since the coordinates of P and Q, which are points on the line, must both satisfy the equation, we have

$$A(x + \Delta x) + B(y + \Delta y) + C = 0;$$

whence $$Ax + A\Delta x + By + B\Delta y + C = 0.$$

By subtracting $$Ax + By + C = 0,$$

we then have $$A\Delta x + B\Delta y = 0,$$

whence $$\frac{\Delta y}{\Delta x} = -\frac{A}{B},$$

or, from page 174, $$m = -\frac{A}{B}.$$

Exercises. Equations of the First Degree

In each of the following cases draw the locus by first locating two convenient points through which it passes:

1. $y + 2x = 7$.
2. $2y - x + 9 = 0$.
3. $2y + 4x = 18$.
4. $7y - 9x = 63$.
5. $18y + 14x = 7$.
6. $0.05x + 0.07y = 0.35$.
7. $0.6x - 1.2y = 9$.
8. $x = 2.75y$.
9. $\frac{2}{3}x - \frac{1}{3} = 4y$.
10. $1.75x + 2.25y = 4.50$.

11. In Exs. 1–10 determine by inspection the slope of the locus of each of the given equations, writing the results.

12. In Exs. 1–10 select the pairs of parallel lines and state the reason for believing them parallel.

13. In Exs. 1–10 select the pairs of perpendicular lines and state the reason for believing them perpendicular.

14. In each of Exs. 1–10 assume two points on the locus which have the coordinates (x, y) and $(x + \Delta x, y + \Delta y)$. Find the slope by finding the difference quotient for the coordinates of these points.

For example, since the assumed points are points on the locus, we may substitute $x + \Delta x$ for x and $y + \Delta y$ for y in Ex. 1, and have

$$y + \Delta y + 2(x + \Delta x) = 7,$$

whence $$y + \Delta y + 2x + 2\,\Delta x = 7.$$

By subtracting $$y + 2x = 7,$$

we then have $$\Delta y + 2\,\Delta x = 0;$$

whence $$\frac{\Delta y}{\Delta x} = -2,$$

or $$m = -2.$$

15. Find the equations of two straight lines which pass through the point $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$ and have inclinations of 60° and 120° respectively. What kind of triangle do the lines make with the x axis? Prove your answer.

In each of the following cases construct the straight line which passes through the given point and has the difference quotient specified:

16. $(1, 2)$; $\frac{3}{4}$. 18. $(2, -4)$; -2. 20. $(\sqrt{2}, 1)$; 1.

17. $(-1, 3)$; 3. 19. $(-2, -3)$; $-\frac{2}{3}$. 21. $(5, 1)$; $\sqrt{3}$.

22. Find the equation of the straight line which passes through the point (3, 4) and cuts off equal positive intercepts on the two axes.

23. If in Ex. 22 we disregard the signs of the intercepts, is more than one solution possible? If so, consider the other case or cases.

24. An automobile starting from a point 15 mi. north of a place A proceeds directly north at a speed of 20 mi./hr. Write the equation which expresses the value of d, the distance that the car is north of A at the end of t hours. Construct the locus of the equation and from it find the approximate value of d when $t = 2\frac{3}{4}$. Show that the rate of the car and the slope of the locus are the same, and interpret the meaning of the negative t intercept.

25. An article describing a proposed income-tax law is illustrated by a straight-line graph which shows that on incomes of \$2000 or less no tax is to be paid, while on \$5000 the tax is \$120. Write the equation which shows the value of t, the amount of tax, in terms of i, the size of the income. By comparing the equation with the formula $y = mx + b$, find the rate r (the per cent) at which the tax is computed.

26. Taking i on the x axis with one space for each \$500, and taking t on the y axis with one space for each \$10, reconstruct the graph of Ex. 25, and from it find the taxes on incomes of \$3000, \$4500, and \$6000.

Taking different scales on the axes changes the slope of the locus. If the spaces are properly interpreted, however, this has no effect here.

27. Construct the loci of the equations $y = 3x - 2$, $y = 3x$, and $y = 3x + 2$. What general relation do the lines have? How could we have shown this relation by simply comparing the equations with the formula $y = mx + b$?

28. The truth of the law $V_t = V_0(1 + 0.00367t)$, where V_t is the volume of a certain gas at t degrees centigrade and V_0 is the volume at 0° C., was proved by laboratory experiments. Construct the graph of this equation when $V_0 = 100$ cc. (cubic centimeters), and from it find the approximate volume at $-10°$ C. of this quantity of the gas.

It is left to the student to select convenient scales on the axes. It should be remembered, however, that this changes the slope of the line.

29. When a certain kind of brass wire is placed under tension, the length changes according to the formula $l = 0.000087\,p + l_0$, where l is the length under a tension of p pounds and l_0 is the original length. If the locus is constructed, what slope will it have? Between what limits will the inclination of the locus lie?

30. The electric resistance r per meter of a copper wire 1 mm. in diameter at a temperature of t degrees centigrade is given by the formula $r = 0.0203 \cdot (1 + 0.0041\,t)$. What is the slope of the locus of this equation? Using convenient scales on the axes, construct the locus.

31. Find the equations of the diagonals of the rectangle in which three vertices are $(-3, 1)$, $(-2, 4)$, and $(4, 2)$.

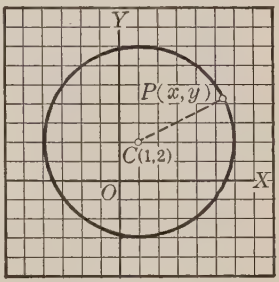
32. Find the equations of the diagonals of the parallelogram in which the sides are represented by the equations $x - a = 0$, $x - b = 0$, $y - c = 0$, and $y - d = 0$.

33. Are the points $(4, 6)$ and $(6, 8)$ on the same or on opposite sides of the line $2x - y + 2 = 0$? How do the slope and the intercepts of the line through these two points compare with those of the given line?

4. Circular Loci

The Circle as a Locus. If we consider loci in the order of simplicity, the next one after the straight line is the circle. From plane geometry it is known that a circle is determined if its center and the length of the radius are given. Although a circle may be determined by other conditions, we shall first consider a special case of finding the equation of a circle when the center and the radius are given.

For example, in finding the equation of the circle generated by a point which moves so as to be always 5 units from the given point $C(1, 2)$, we may first let $P(x, y)$ be any point on the locus.



As suggested on page 178, we now seek to formulate some statement regarding the coordinates of P. From the given conditions we have

$$CP = 5,$$

and by using the formula for l, the length of a line segment joining two points (page 168), we have

$$l = CP = \sqrt{(x-1)^2 + (y-2)^2}.$$

Hence $$\sqrt{(x-1)^2 + (y-2)^2} = 5,$$

or $$(x-1)^2 + (y-2)^2 = 25,$$

which is the equation of the circle that is generated by a point moving in accordance with the given conditions.

Such an equation may, if desired, be written in other forms; for example, if we expand the equation of the circle that we found above, simplify, and rearrange the terms, we have

$$x^2 + y^2 - 2x - 4y - 20 = 0.$$

At present, however, the form $(x-1)^2 + (y-2)^2 = 25$ is better suited to our purposes.

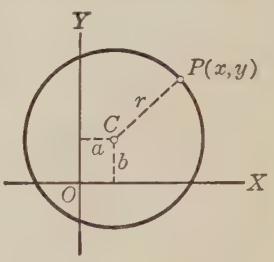
General Equation of a Circle. In considering the general equation of the circle when the center and radius are given, we may proceed in the same way as in the special case considered on page 195.

That is, letting a, b, and r represent real numbers, we may take $C(a, b)$ in the figure at the right as the given center of the circle and r as the given radius. We then have, as before,

$$CP = r,$$

and

$$r = \sqrt{(x-a)^2 + (y-b)^2}.$$



Squaring, we then have

$$(x-a)^2 + (y-b)^2 = r^2,$$

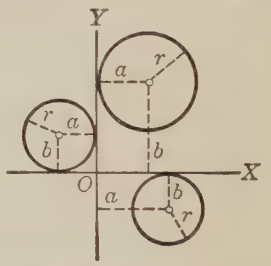
which is the general equation of the circle with the radius r and the center at the point (a, b).

Circles Tangent to an Axis. If a circle is tangent to the y axis, as shown in the figure below, we see that a, the abscissa of the center, has the same numerical value as r. That is, since $a^2 = (-a)^2 = r^2$, the equation of such a circle is

$$(x-a)^2 + (y-b)^2 = a^2.$$

Similarly, if the circle is tangent to the x axis, $b^2 = (-b)^2 = r^2$, and hence its equation is

$$(x-a)^2 + (y-b)^2 = b^2.$$



The symbol $|a|$ is used for the numerical, or absolute, value of a; that is, for the value without regard to sign. Thus, if a circle is tangent to both axes, as in the figure above, $|a| = |b| = r$, and conversely.

From these three cases, we see that if we are given the center of a circle which is tangent to either one or to both of the axes, we can find its equation.

Discussion of the General Equation. If a, b, and r, or some condition relating to them, are not given by the conditions which determine the circle, it is evident that the equation

$$(x-a)^2+(y-b)^2=r^2$$

is not convenient in writing the equation of the locus. The above equation, however, may be written as

$$x^2+y^2-2ax-2by+a^2+b^2-r^2=0,$$

in which we see that the coefficients of x^2 and y^2 are both 1, that there is a term $(-2ax)$ in x and one $(-2by)$ in y, and that the rest of the left-hand member $(a^2+b^2-r^2)$ may be regarded as a single term that is independent of both x and y.

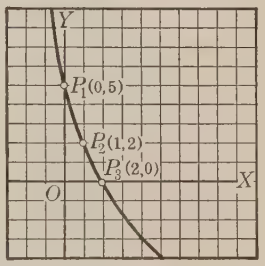
We may therefore write the equation of the circle as

$$x^2+y^2+Dx+Ey+F=0,$$

where the significance of the three arbitrary constants D, E, and F will be considered later.

The case in which this equation does not represent a circle will be considered on page 201.

Circle through Three Points. The above formula enables us to write the equation of the circle through three given points not in a straight line. For example, if the three points are $P_1(0, 5)$, $P_2(1, 2)$, and $P_3(2, 0)$, we may substitute in the formula and have



$$0+25+0+5E+F=0,$$

$$1+4+D+2E+F=0,$$

and $$4+0+2D+0+F=0.$$

Solving these three simultaneous equations, we then have $D=-37$, $E=-19$, and $F=70$, and hence the equation is

$$x^2+y^2-37x-19y+70=0.$$

The method of constructing the circle will be considered on page 200.

Exercises. Circular Loci

1. Find the equation of the locus of a point which moves so that it is always 3 units from the point (1, 3).

In each of the following cases find the equation of the circle in which the center and the length of the radius are as follows:

2. (3, − 2), 5.
3. (− 4, 5), 9.
4. (− 1, − 3), $\frac{1}{4}$.
5. (0, 0), 7.
6. (6, 0), 4.
7. (0, 4), 5.
8. $(-\sqrt{2}, 1)$, 2.
9. $(-\frac{1}{2}, -\frac{3}{4})$, 4.
10. $(\sqrt{2}, \sqrt{3})$, $\sqrt{6}$.

11. Find the equation of the circle which is tangent to the x axis and has the point (3, − 5) for its center.

12. Find the equation of the circle which is tangent to the y axis and has the point (− 1, − 2) for its center.

13. Find the equations of the four circles which are tangent to both axes and have a radius of 4 units.

14. Find the equation of the circle which can be constructed upon the line segment joining the points (− 2, − 3) and (6, − 5) as diameter.

15. Find the equations of the two circles which have a radius of 5 units and in which the line segment joining the points (1, 2) and (3, 4) is a common chord.

In each of the following cases find the equation of the circle determined by the given points:

16. (0, 0), (3, 0), (0, − 4).
17. (0, 2), (0, − 2), (6, 0).
18. (1, 3), (4, − 1), (3, − 6).
19. (2, 3), (− 3, 2), (1, − 2).

20. Find the equation of the circle circumscribed about the triangle for which the equations of the three sides are $2x - y = 9$, $x + y = 6$, and $3x + y = 6$.

First solve the linear equations and thus find the three points which determine the circle.

General Equation of the Second Degree. As we have already learned (page 189), the most general equation of the first degree in x and y has the form

$$Ax + By + C = 0.$$

The most general equation of the second degree in x and y must contain terms in x^2, y^2, xy, x, and y, and must also contain some term, called the *constant* (or *absolute*) *term*, that is independent of both x and y. In other words, such an equation must contain three terms of the second degree, which have x^2, y^2, and xy as the literal factors, two of the first degree, and one of the zero degree. It is therefore apparent that the general equation has the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

The General Equation and the Circle. At present we shall not consider the locus of the general equation of the second degree given above, but shall take only the special case in which $a = c$ and $b = 0$. If we substitute these values for a, c, and b, the equation becomes

$$ax^2 + ay^2 + dx + ey + f = 0,$$

or

$$x^2 + y^2 + \frac{d}{a}x + \frac{e}{a}y + \frac{f}{a} = 0.$$

Writing D for $\frac{d}{a}$, E for $\frac{e}{a}$, and F for $\frac{f}{a}$, in order to avoid fractions, we then have

$$x^2 + y^2 + Dx + Ey + F = 0.$$

As was shown on page 197, an equation of the second degree in this form is the equation of a circle.

We shall now show how to find the coordinates of the center and the length of the radius of the circle which this equation represents. When we have found these, it is evident that we can construct the locus mechanically with a pair of ordinary compasses.

Finding the Center and Radius. Taking (a, b) as the center and r as the radius, the general equation of a circle is

$$(x-a)^2+(y-b)^2=r^2,$$

or
$$x^2+y^2-2ax-2by+a^2+b^2-r^2=0.$$

If this is now compared with the other general equation,

$$x^2+y^2+Dx+Ey+F=0,$$

it is evident that we have the following relations:

$$-2a=D, \qquad -2b=E, \qquad a^2+b^2-r^2=F,$$

or
$$a=-\tfrac{1}{2}D, \qquad b=-\tfrac{1}{2}E, \qquad r^2=a^2+b^2-F.$$

By applying these relations, we then see that

$$r^2=\tfrac{1}{4}D^2+\tfrac{1}{4}E^2-F,$$

or
$$r=\tfrac{1}{2}\sqrt{D^2+E^2-4F}.$$

Although there are certain exceptions, which will be considered on page 201, we may now make the following summary regarding the equation $x^2+y^2+Dx+Ey+F=0$:

1. *The locus is a circle.*

2. *The center of this circle is the point* $(-\tfrac{1}{2}D, -\tfrac{1}{2}E)$; *that is, the abscissa is half the coefficient of* x *with its sign changed, and the ordinate is half the coefficient of* y *with its sign changed.*

3. *The length of the radius is half the square root of the sum of the squares of the coefficients of* x *and* y *decreased by four times the absolute term.*

For example, if we divide both members by 4, the equation

$$4x^2+4y^2+8x-12y-3=0$$

becomes
$$x^2+y^2+2x-3y-\tfrac{3}{4}=0.$$

Also, $r=\tfrac{1}{2}\sqrt{D^2+E^2-4F}=\tfrac{1}{2}\sqrt{4+9+3}=\tfrac{1}{2}\cdot 4=2.$

That is, the locus of the given equation is the circle with the center $(-1, \tfrac{3}{2})$ and the radius 2. This locus is easily constructed by first locating the center and then using the compasses.

Existence of a Circular Locus. Although we have shown (page 190) that every equation of the first degree in x and y has a straight line for its locus, it is not true that every second-degree equation of the form

$$x^2 + y^2 + Dx + Ey + F = 0$$

has a circle for its locus. It is evident that if we are to plot a curve by finding the coordinates which satisfy the equation, x and y must both be real numbers. But when $D = E = 0$ it is possible to have an equation of the second degree in x and y that is satisfied by no real values of x and y.

Such an equation is $x^2 + y^2 + 4 = 0$, which has imaginary roots. It is therefore impossible to construct the locus of this equation. We cannot at this time discuss the geometric representation of imaginary numbers, but it would not affect the statements with regard to a real circle.

Whether or not there is a locus for the above equation depends upon the value of $D^2 + E^2 - 4F$ in the formula

$$r = \tfrac{1}{2}\sqrt{D^2 + E^2 - 4F}.$$

An examination of this formula shows that we have the following possibilities:

1. $D^2 + E^2 - 4F > 0$. That is, if $D^2 + E^2 - 4F$ is positive, the value of r is real, and hence a locus exists and the circle can be constructed.

This is seen to be the case in $x^2 + y^2 - 4x + 2y - 15 = 0$.

2. $D^2 + E^2 - 4F = 0$. In this case we have $r = 0$, and since the center is the point $(-\frac{1}{2}D, -\frac{1}{2}E)$, as shown on page 200, the locus consists of a single point, the center.

This is seen to be the case in $x^2 + y^2 + 2x - 3y + 1\frac{3}{4} = 0$.

3. $D^2 + E^2 - 4F < 0$. That is, if $D^2 + E^2 - 4F$ is negative, the value of r is imaginary, and hence no circle can be constructed.

This is seen to be the case in $x^2 + y^2 + 2x - 3y + 10 = 0$.

Exercises. Circular Loci

Find the length of the radius and the coordinates of the center of the circle represented by each of the following equations:

1. $x^2 + y^2 - x - y = 0$.

2. $x^2 + y^2 - 3x - 2y + 2 = 0$.

3. $4x^2 + 4y^2 + 4x - 8y + 3 = 0$.

4. $2x^2 + 2y^2 + 4x + 3y + 1 = 0$.

In each of the following cases construct the locus represented by the equation, or state why it is impossible to do so:

5. $x^2 + y^2 + 2x + y + 2 = 0$.

6. $x^2 + y^2 - 7x - 3y + 3 = 0$.

7. $2x^2 + 2y^2 - 14x + 6y + 5 = 0$.

8. $3x^2 + 3y^2 - 6y + 4 = 0$.

9. Prove that $(x - a)(x - a') + (y - b)(y - b') = 0$ is the equation of a circle in which the coordinates of the ends of a diameter are (a, b) and (a', b') respectively.

10. Find the equation of the circle which is tangent to the y axis and has the center $(2, 4)$; which is tangent to the x axis at the origin and passes through $(0, -6)$.

11. Find the equations of all the circles which are tangent to both axes and have a radius of $\frac{1}{2}\sqrt{2}$ units.

12. Find the equation of a circle which has its center at the point $(1, -1)$ and a radius of 6.5 units. Through which of the points $(-1.5, 5)$, $(7, 1.5)$, $(1.5, 5)$, $(1, 5.5)$, and $(1, -7.5)$ does the circle pass? Construct the locus and locate the points to verify your answers.

13. Find the equation of the circle circumscribed about a square which is s units on a side and has its vertices at the points $(0, 0)$, $(s, 0)$, (s, s), and $(0, s)$.

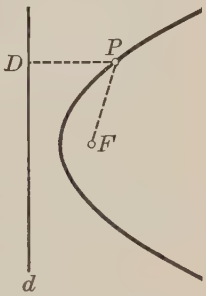
5. The Parabola

Other Loci of the Second Degree. In discussing the circle as a locus, it was stated that not every equation of the second degree represents a circle; in fact, we found that only equations of the form

$$x^2 + y^2 + Dx + Ey + F = 0$$

have circles for their loci. We shall now consider other locus problems which lead to equations of the second degree in x and y.

Parabola. Just as a circle is defined as the locus of a point which moves so as always to be at a given distance from a fixed point in the plane, so the locus of a point which moves so as always to be equidistant from a given point and a given line in the plane is called a *parabola.*



For example, this figure shows a parabola generated by the point P, which moves so that it is always equidistant from the point F and the line d; that is, so that $FP = DP$.

From the figure and the definition it is evident that a parabola is an open curve which may be extended indefinitely.

The given point (F in the figure) is called the *focus* of the parabola, and the given line (d in the figure) is called the *directrix* of the curve.

The parabola is a curve with important practical applications. Automobile headlights, searchlights, and so on are generally provided with reflectors which have a parabolic cross section. If a lamp is placed at the focus of such a reflector, the rays of light are reflected in a beam perpendicular to the directrix.

The parabola is also spoken of as one of the *conic sections*, because it is one of the several different curves that is formed by the intersection of a plane with a right circular cone. In our work we shall use only analytic methods, and hence the geometric conception is not important.

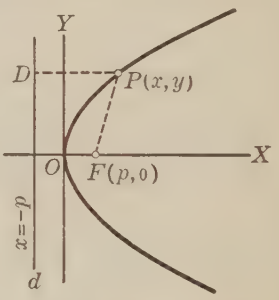
A Special Case of the Parabola. Although the focus of a parabola may be any point in the plane and the directrix any line, we shall find that it simplifies the work to make a special choice of both directrix and focus. For the purpose of developing the equation of the locus under these conditions, we shall therefore consider only the case in which the focus of the parabola is a point on the x axis and the directrix is parallel to the y axis.

In some of the exercises, as on page 206, we shall consider the parabola which has its focus on the y axis and its directrix parallel to the x axis, and we shall also consider more general cases.

Simplest Equation of the Parabola. If in this figure we let the focus be $F(p, 0)$, a point on the x axis to the right of the origin O, and let the directrix d be parallel to the y axis and p units to the left of O, we see that for any point $P(x, y)$ of the locus

$$FP = \sqrt{(x-p)^2 + (y-0)^2},$$

and $$DP = p + x.$$



From the definition of a parabola (page 203) we then have

$$FP = DP,$$

or $$\sqrt{(x-p)^2 + y^2} = x + p.$$

Hence, by squaring and simplifying this last equation, we have

$$y^2 = 4px,$$

which is the equation of the parabola with the focus $(p, 0)$ and the directrix $x = -p$.

By making this special choice of focus and directrix we obtain the equation of the parabola in its simplest form. Since this choice brought the focus and directrix at equal distances on opposite sides of the y axis, the origin O is a point on the curve. This is also evident from the fact that when $x = 0$ in the equation $y^2 = 4px$, we have $y = 0$.

The equations of the more general cases of the parabola are easily developed by the method given above.

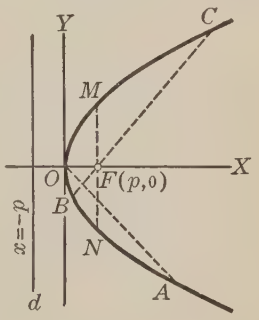
Parts of a Parabola. Besides the focus and directrix, there are certain other parts of a parabola to which we give special names. For example, the line through the focus perpendicular to the directrix is called the *axis* of the parabola.

When we take the focus on the x axis, the axis of the parabola coincides with the x axis, as shown in this figure.

The point where the axis cuts the parabola is called the *vertex*.

In the parabola $y^2 = 4\,px$ the vertex is the origin; that is, it is the point (0, 0).

A line joining any two points on the curve is called a *chord*; a chord passing through the focus is called a *focal chord*; and a line from the focus to the curve is called a *focal radius*.



In the figure, BC and MN are focal chords, and FC is a focal radius.

The length of the focal chord perpendicular to the axis, as MN in the figure, is called the *focal width*. Since $OF = p$, if we substitute p for x in the equation $y^2 = 4\,px$, we see that $FM = \sqrt{4\,p^2} = 2\,p$, and hence for the focal width w_f we have

$$w_f = 4\,p.$$

The Latin name *latus rectum* is also used for the *focal width*.

Exercises. The Parabola

1. Find the equation of the locus of a point which is always equidistant from the point $(-\,p,\,0)$ and the line $x = p$.

2. Find the equation of the locus of a point which is always equidistant from the point $(0,\,p)$ and the line $y = -\,p$.

3. Find the equation of the parabola in which the focus is the point $(0,\,-\,p)$ and the directrix is the line $y = p$.

4. Find the equation of the locus of a point which is always equidistant from the point $(2,\,3)$ and the line $x = -\,5$.

Write the equation of the parabola in which the focus and directrix are respectively as follows:

5. $(2, 0)$, $x = -2$.
6. $(-3, 0)$, $x = 3$.
7. $(5, 0)$, $x = -5$.
8. $(\frac{1}{2}, 0)$, $x = -\frac{1}{2}$.
9. $(-\frac{5}{4}, 0)$, $x = \frac{5}{4}$.
10. $(0, 2)$, $y = -2$.
11. $(0, -3)$, $y = 3$.
12. $(0, a)$, $y = -a$.
13. $(0, -a)$, $y = a$.
14. $(-1, 0)$, $x = 2$.
15. $(5, 1)$, $x = -3$.
16. $(3, -4)$, $y = 2$.

17. Find the equation of the parabola in which the directrix is the line $y = 6$ and the vertex is the point $(0, 0)$.

18. Solve Ex. 17 when the directrix is the line $y = -8$.

19. Find the equation of the parabola in which the vertex is $(0, 0)$, the focus is on the x axis, and the focal width is 8.

20. Find the equation of the parabola which has its focus at the origin and its vertex at the point $(2, 0)$.

21. Find the equation of the parabola which has its vertex at the origin and its axis along the x axis, and which passes through the point $(2, 3)$.

22. Solve Ex. 21 for the parabola which passes through the point $(-1, -5)$; the point $(-2, 3)$; the point $(3, -4)$.

23. Show that the distance from the focus to any point $P_1(x_1, y_1)$ on the parabola $y^2 = 4px$ is $x_1 + p$.

24. Find the equation of the locus of a point which moves so as always to be equidistant from the point $(a, 0)$ and the line $x = -p$.

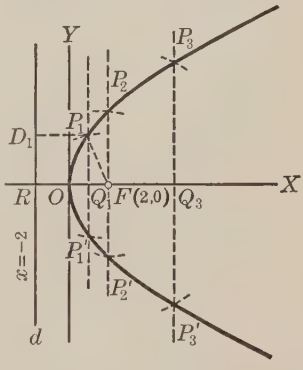
25. In Ex. 24 show that the curve is a parabola with the directrix $x = -p$ and the focus $(a, 0)$, and find the coordinates of the vertex and the focal width.

26. Show that the locus of a point which moves so as to be equidistant from the point (a, b) and the line $x = -p$ is a parabola, and find the equation of the curve, the coordinates of the vertex, and the focal width.

Constructing a Parabola. If the equation of a parabola is of the form $y^2 = 4\,px$, or the form $x^2 = 4\,py$, the curve can be easily constructed. To illustrate the construction let us take the equation $y^2 = 8\,x$.

In this equation, which is of the form $y^2 = 4\,px$, we see that $4\,p = 8$ and hence $p = 2$. That is, the directrix is the line $x = -2$, the focus is the point (2, 0), the focal width is 8, and the vertex is the point (0, 0).

Drawing a pair of x and y axes, as shown in the figure below, we then draw the directrix d parallel to the y axis and 2 units to the left of O, and we locate the focus F on the x axis 2 units to the right of O.



In such cases we let some convenient length, such as $\frac{1}{8}$ in., $\frac{1}{4}$ in., or 0.1 in., be the unit by which we measure.

Drawing a line through F perpendicular to the x axis, we then lay off FP_2 and FP_2' in such a way that $FP_2 = FP_2' = 4$, and hence the focal width P_2P_2' is 8.

We have now located three points, O, P_2, and P_2', which are all on the required curve.

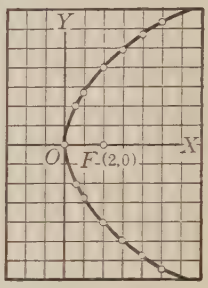
To find other points, we may take any point Q_1 on the x axis, and through it draw a line perpendicular to OX. Then taking RQ_1 (the distance to Q_1 from the intersection of the directrix and the x axis) as a radius, and F as a center, we may locate the points P_1 and P_1' with the compasses.

Since $D_1P_1 = RQ_1 = FP_1$, we see that P_1 is a point on the parabola, and similarly for P_1'.

Other pairs of points, like P_3 and P_3', may be located in the same way, and through them we may draw a smooth curve, which is the required parabola.

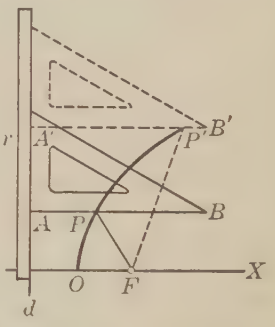
Graph of a Parabola. For practical purposes the simplest method of locating points on a parabola is the familiar one of plotting its graph. Thus, in the equation $y^2 = 8x$, which was considered on page 207, we may assign convenient values to x and compute the corresponding values of y, as shown in the following table:

If $x =$	0	0.5	1	2	3	4	5	···
$y =$	0	±2	±2.8	±4	±4.9	±5.7	±6.3	···



Locating these points in the usual way, we may then draw the parabola through them, as shown in the figure at the right.

Mechanical Construction. The parabola may also be constructed mechanically, after the directrix and focus have been located, by using a ruler, a draftsman's right triangle, and a piece of thread. The ruler r is placed along the directrix d and the triangle is placed with one side against the ruler, as here shown. One end of a thread equal in length to the side AB of the triangle is then fastened to a pin at the focus F, and the other end is fastened or held at the vertex B. A pencil point P is then held against the side AB of the triangle so as to hold the thread taut. If the triangle is now allowed to slide along the ruler, as shown by the dotted lines in the figure, the pencil point will trace a portion of a parabola. The part of the curve below the x axis can then be constructed by changing the position of the triangle and ruler.



From the figure we see that $FP + PB = AB$ and $AP + PB = AB$, and hence $FP = AP$; that is, the point P satisfies the definition of a parabola, and similarly for P' or any other point determined in this way.

Exercises. The Parabola

Construct each of the following parabolas, using the method given on page 207 for the first two, the graphic method given on page 208 for the next two, and the mechanical method (if instruments are available) for the last two:

1. $y^2 = 2x$. **3.** $y^2 = 12x$. **5.** $x^2 = 6y$.

2. $x^2 = 2y$. **4.** $x^2 = -12y$. **6.** $y^2 = -6x$.

7. For the parabola $y^2 + 4x + 8 = 0$ find the equation of the directrix and the coordinates of the focus and vertex, and construct the parabola.

The given equation is of the form $y^2 - 2(a + p)x + a^2 - p^2 = 0$, which was found in Exs. 24 and 25 on page 206 to be the equation of the parabola with the directrix $x = -p$ and the focus $(a, 0)$. Then $a^2 - p^2 = 8$ and $-2(a + p) = 4$, and from these two relations the equation of the directrix and the coordinates of the focus and vertex are easily found.

In each of the following cases find the equation of the directrix and the coordinates of the focus and vertex, and construct the parabola by the most convenient method:

8. $y^2 + 16x + 24 = 0$. **10.** $x^2 - 4y - 6 = 0$.

9. $y^2 - 8x + 32 = 0$. **11.** $x^2 - 10y + 50 = 0$.

12. For the parabola $y^2 - 4y - 8x + 12 = 0$ find the equation of the directrix and the coordinates of the focus and vertex, and construct the parabola.

In the form $(y - b)^2 - 2(a + p)x + a^2 - p^2 = 0$, which was found in Ex. 26 on page 206 to be the equation of the parabola with the directrix $x = -p$ and the focus (a, b), the given equation is $(y - 2)^2 - 2 \cdot 4x + 8 = 0$; that is, $b = 2$, $a + p = 4$ and $a^2 - p^2 = 8$. From these relations the solution is then easily completed.

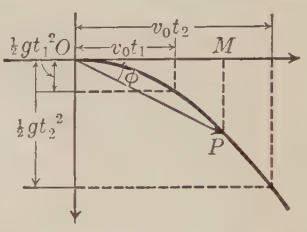
Solve Ex. 12 for each of the following equations:

13. $y^2 + 4y - 4x + 4 = 0$. **16.** $y^2 - 4y - 6x + 10 = 0$.

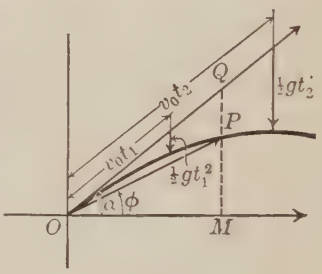
14. $y^2 - 8y + 4x + 16 = 0$. **17.** $y^2 + 8y + 6x + 24 = 0$.

15. $y^2 - 2y - 4x + 17 = 0$. **18.** $y^2 - 6y + 8x + 24 = 0$.

19. In Ex. 16 on page 159 it was shown that, when a body is projected horizontally from a point O above the ground, the horizontal and vertical components at the end of t seconds are given by the equations $x = v_0 t$ and $y = -\frac{1}{2} gt^2$. By eliminating t, we have $x^2 = -\dfrac{2 v_0^2 y}{g}$, which is the equation of a parabola. Taking the approximate value 32 for g and a velocity v_0 of 8 ft./sec., find the equation of the directrix and the coordinates of the focus, and construct the parabola.



20. In Ex. 19 on page 160 it was shown that the horizontal and vertical components of the path of a projectile fired at an angle α with the horizontal are represented by $x = v_0 t \cos \alpha$ and $y = v_0 t \sin \alpha - \frac{1}{2} gt^2$. Taking $g = 32$, $\alpha = 45°$, and $v_0 = 8$ ft./sec., show that the equation obtained by eliminating t is the equation of a parabola, find the equation of the directrix and the coordinates of the focus and vertex, and construct the parabola.



21. Write the equation of the parabola in which the directrix is the y axis and the focus is on the x axis.

22. Find the common points of the graphs of the equations $y^2 = 2x$ and $x^2 = 2y$, and thus solve the equations.

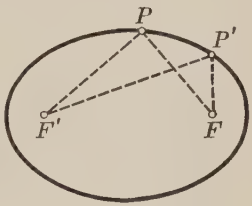
23. Solve the equations $x^2 + y = 7$ and $x + y^2 = 11$ by drawing the parabolas which they represent.

24. Find the equation of the parabola in which the vertex is at the origin, the focus is on the y axis, and the line joining the points (2, 6) and $(-2, 6)$ is a chord.

6. The Ellipse

Ellipse. In our work thus far we have considered two important curvilinear loci: (1) the circle, which is the locus of a point that moves so as always to be equidistant from a fixed point in the plane, and (2) the parabola, which is the locus of a point that moves so as always to be equidistant from a fixed point and a fixed line in the plane.

We shall now consider a third important locus, the *ellipse*, which is the locus of a point that moves so that the sum of its distances from two fixed points in the plane is constant.



In this figure the two fixed points are F and F', and the ellipse is traced by the point P which moves so that $FP + F'P$ is constant; that is, if we designate the constant sum by $2\,a$, the equation that expresses the condition under which the locus of P is an ellipse is that

$$FP + F'P = 2\,a.$$

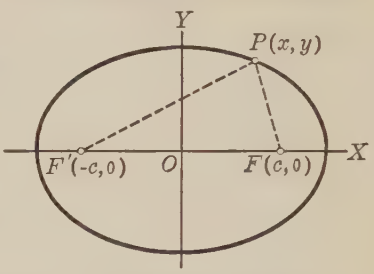
The two fixed points, F and F', are called the *foci* of the ellipse, and the lines FP and $F'P$ are called the *focal radii* from the point P.

Mechanical Construction. By applying the above definition we can construct an ellipse mechanically by using two pins and a piece of thread or string.

The pins are first placed at the foci, F and F', and the thread is then fastened to them in such a way that the length of the thread between the pins is equal to the desired constant sum. The thread is then stretched taut with a pencil point P, and by moving the pencil, as from P to P' in the above figure, a portion of the ellipse is traced.

The student is advised to try this method of constructing an ellipse, and to observe that as the foci are allowed to approach each other, the ellipse approaches a circle. We may therefore consider a circle as a special kind of ellipse, — the one in which the two foci coincide.

Standard Equation of the Ellipse. By taking the two foci at the points $F(c, 0)$ and $F'(-c, 0)$, which are points on the x axis, and by letting $P(x, y)$ be any point of the locus, we can easily find the equation of the ellipse in which the constant sum of the focal radii from P is $2a$.



That is, the origin is taken halfway between the foci.

From the formula for l given on page 168 we first have

$$FP = \sqrt{(x-c)^2 + (y-0)^2},$$

and
$$F'P = \sqrt{(x+c)^2 + (y-0)^2};$$

whence
$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a.$$

Transposing the second radical, squaring, and simplifying, we then have

$$a\sqrt{(x+c)^2 + y^2} = a^2 + cx,$$

from which, by squaring again and simplifying, we have

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2),$$

or
$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

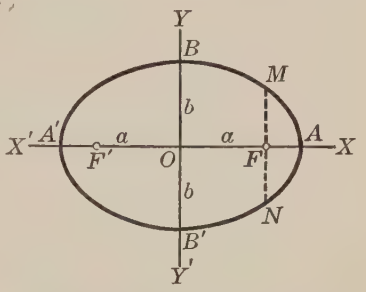
Since $FP + F'P > 2c$, we see that $a > c$, and hence $a^2 - c^2$ is always positive. If we now introduce a new constant b such that $b^2 = a^2 - c^2$, the above equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is known as the *standard equation* of the ellipse in which the foci are the points $(c, 0)$ and $(-c, 0)$, the constant sum is $2a$, and $b^2 = a^2 - c^2$.

If any two of the three constants a, b, and c are given, the third may be found from the relation $b^2 = a^2 - c^2$, and hence the ellipse is determined.

Parts of an Ellipse. As with the parabola, there are certain lines and points of an ellipse to which it is convenient to apply special names. For example, the point which is halfway between the foci, as O in this figure, is called the *center* of the ellipse.



The line through the foci and the line perpendicular to this line at O are called the *axes* of the ellipse.

When the foci are taken as on page 212, the center is the origin, and the axes of the ellipse are the x and y axes.

The points A and A' where the ellipse cuts the axis through F and F' are called the *vertices* of the ellipse.

The segment AA' of the axis through the foci is called the *major axis*, and the segment BB' is called the *minor axis*.

If $y = 0$ in the standard equation, we have $x = a$; that is, the length of the major axis is $2a$. Similarly, the length of the minor axis is $2b$, and hence a and b in the equation represent the major and minor semiaxes.

The length of the chord drawn through either focus perpendicular to the major axis is called the *focal width*.

The Latin name *latus rectum* is also used. In the figure, MN represents the focal width, but the chord could as well have been drawn through F'.

For the point M in the above figure we see that $x = c$, and hence, substituting c for x in the standard equation, we have

$$\frac{c^2}{a^2} + \frac{y^2}{b^2} = 1;$$

whence

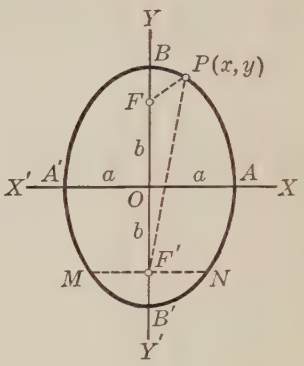
$$y = \sqrt{\frac{b^2(a^2 - c^2)}{a^2}}.$$

Then since $a^2 - c^2 = b^2$ and the focal width w_f is evidently equal to $2y$, we have

$$w_f = \frac{2b^2}{a}.$$

Foci on the y Axis. If the foci are taken on the y axis at the points $F(0, c)$ and $F'(0, -c)$, as here shown, the equation of the ellipse in which the constant sum of the focal radii is $2b$ is the same as before; namely,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



In this case, however, a is the minor semiaxis, b is the major semiaxis, $a^2 = b^2 - c^2$, and for the focal width w_f, or MN, we have

$$w_f = \frac{2a^2}{b}.$$

Exercises. The Ellipse

1. By first finding the values of a and b, write the equation of the ellipse in which the foci are at $(2, 0)$ and $(-2, 0)$ and the constant sum is 6, and then construct the locus.

In each of Exs. 2–5, below, find the equation and construct the ellipse which is determined as follows:

2. Foci, $(5, 0)$ and $(-5, 0)$; major axis, 20.

3. Vertices, $(12, 0)$ and $(-12, 0)$; minor axis, 10.

4. Foci, $(0, 2\sqrt{3})$ and $(0, -2\sqrt{3})$; minor axis, 12.

5. Major axis, 10; minor axis, $2\sqrt{15}$; foci on the x axis.

6. Find the major and minor axes, the coordinates of the foci and vertices, and the focal width of the ellipse which is represented by the equation $\frac{1}{16}x^2 + \frac{1}{4}y^2 = 1$.

Solve Ex. 6 for each of the following equations:

7. $4x^2 + y^2 = 16$.

8. $3x^2 + 4y^2 = 48$.

9. $7x^2 + 4y^2 = 28$.

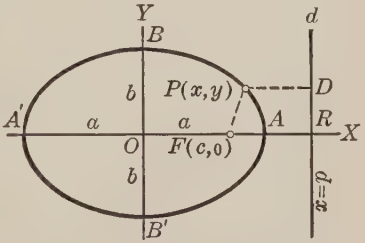
10. $3x^2 + 5y^2 = 15$.

Boscovich's Definition. Boscovich (an Italian mathematician, about 1750) defined the ellipse as the locus of a point which moves so that the ratio of its distance from a given point (the *focus*) to its distance from a given line (the *directrix*) is a constant less than 1.

That is, representing this constant (called the *eccentricity*) by e, the condition that $P(x, y)$ in the figure below generates an ellipse is that

$$\frac{FP}{PD} = e < 1.$$

If, as before, we take a as the major semiaxis and $F(c, 0)$ as the focus of the ellipse and take the line $x = p$ as the directrix, we have



$$\frac{FA}{AR} = \frac{a-c}{p-a} = e,$$

and $$\frac{A'F}{A'R} = \frac{a+c}{p+a} = e;$$

or $$a - c = ep - ae,$$

and $$a + c = ep + ae.$$

Adding and subtracting these two equations, we then have

$$a = ep \quad \text{and} \quad c = ae = e^2p.$$

Replacing b^2 by its equal, $a^2 - c^2$, in the standard equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

Hence, in terms of the eccentricity e and the distance p of the directrix from the origin, the equation of the ellipse is

$$\frac{x^2}{e^2p^2} + \frac{y^2}{e^2p^2(1-e^2)} = 1.$$

Since the ellipse is symmetric, it can be equally well developed from the focus $F(-c, 0)$ and the directrix $x = -p$.

Exercises. The Ellipse

1. From the relations $a^2 = e^2p^2$ and $b^2 = e^2p^2(1 - e^2)$ derive a formula for e in terms of a and b, and apply it in finding the eccentricity of the ellipse $\frac{1}{16}x^2 + \frac{1}{4}y^2 = 1$.

2. In Ex. 1 find the equations of the two directrices.

In each of the following cases find the major and minor axes of the ellipse, the coordinates of the foci and vertices, the focal width and eccentricity, and the equations of the directrices:

3. $\frac{x^2}{4} + \frac{y^2}{3} = 1.$

4. $\frac{x^2}{25} + \frac{y^2}{4} = 1.$

5. $2x^2 + 3y^2 = 12.$

6. $3x^2 + 5y^2 = 75.$

7. $7x^2 + 12y^2 = 84.$

8. $x^2 + 4y^2 = 8.$

9. $\frac{y^2}{25} + \frac{x^2}{16} = 1.$

10. $\frac{y^2}{25} + \frac{x^2}{4} = 1.$

11. $5y^2 + 12x^2 = 300.$

12. $y^2 + 2x^2 = 6.$

13. $4y^2 + x^2 = 16.$

14. $y^2 + 5x^2 = 20.$

15. Find the coordinates of the foci of the ellipse represented by the equation $\frac{x^2}{9 + k^2} + \frac{y^2}{4 + k^2} = 1.$

16. Draw the ellipse in Ex. 15 for $k = 1$; for $k = 2$. Do the two ellipses have the same foci?

Construct the ellipses which are determined as follows, and write the equation of the locus below each figure:

17. Foci, (4, 0) and (− 4, 0); major axis, 10.

18. Foci, (0, 3) and (0, − 3); major axis, 12.

19. Vertices, (6, 0) and (− 6, 0); foci, (2, 0) and (− 2, 0).

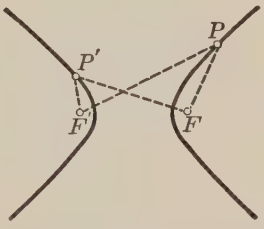
20. Directrices, $x = 5$ and $x = -5$; eccentricity, $\frac{4}{5}$.

21. Foci, (0, 2) and (0, − 2); focal width, $2\frac{2}{3}$.

22. Focal width, $3\frac{3}{5}$; eccentricity, $\frac{4}{5}$; foci on the x axis.

7. The Hyperbola

Hyperbola. We shall now consider another important curvilinear locus called the *hyperbola*. This curve is the locus of a point which moves so that the difference between its distances from two fixed points in the plane is constant.

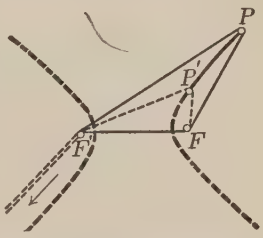


In this figure the two fixed points are F and F', and the curve is traced by the point P, which moves so that the difference $F'P - FP$ is constant. Designating the constant difference by $2\,a$, we see that when $FP > F'P$, as at P' in the figure, $2\,a$ is negative; that is, the condition under which the point P generates a hyperbola is that

$$F'P - FP = \pm 2\,a.$$

The two fixed points are called the *foci*, the lines $F'P$ and FP are called the *focal radii* from P, and the complete hyperbola is said to have two *branches*.

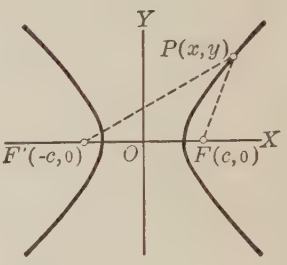
Mechanical Construction. By applying the above definition we can construct a hyperbola mechanically with a piece of thread and two pins.



The two pins are first placed at the foci, F and F'. A pencil point P is then inserted in a loop in the thread, and the thread is placed around the pins, as shown in the figure, in such a way that $F'P - FP = 2\,a$, the desired constant difference. The two ends of the thread are then held together at F' and are drawn in the direction indicated by the arrow. Since the lengths of $F'P$ and FP are reduced equally, $F'P - FP$ remains constant, and hence P traces a portion of a hyperbola, as from P to P'.

By rearranging the thread about the two pins, the extension of the curve, as shown by the heavy dotted lines in the figure, can be constructed in the same way.

Standard Equation of the Hyperbola. By taking the two foci at the points $F(c, 0)$ and $F'(-c, 0)$ on the x axis we can easily find the equation of the hyperbola in which the constant difference between the focal radii is $\pm 2a$.



Taking $P(x, y)$ as any point on the locus, from the formula for l on page 168 we then have

$$F'P = \sqrt{(x+c)^2 + (y-0)^2},$$

and $$FP = \sqrt{(x-c)^2 + (y-0)^2};$$

whence $$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

Transposing the second radical in this equation and squaring both members, we have

$$(x+c)^2 + y^2 = 4a^2 \pm 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2,$$

or $$\pm a\sqrt{(x-c)^2 + y^2} = a^2 - cx.$$

Squaring again and simplifying, we then have

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2),$$

or $$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

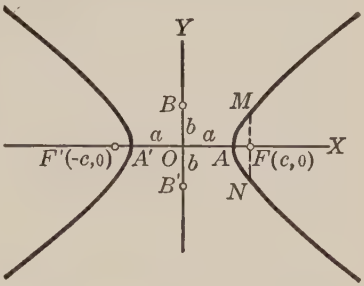
Since $F'P - FP < 2c$, we see that $a < c$, and hence the expression $a^2 - c^2$ is always negative. We may therefore introduce a new constant b such that $b^2 = c^2 - a^2$, and write the term in y^2 with the negative sign; that is, we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

This is known as the *standard equation* of the hyperbola in which the foci are the points $(c, 0)$ and $(-c, 0)$, the constant difference is $\pm 2a$, and $b^2 = c^2 - a^2$.

As with the ellipse (page 212), the third constant may be found when any two of the others are given. It should be observed, however, that for the hyperbola the relation between the constants is $b^2 = c^2 - a^2$, while the corresponding relation for the ellipse is $b^2 = a^2 - c^2$.

Parts of a Hyperbola. The terms *center*, *axes*, and *vertices* are applied to the hyperbola in the same way as they are to the ellipse (page 213).



Thus, in this figure the center is O, the axes (in this case the x and y axes) consist of the line through F and F' and the perpendicular through O, and the vertices are A and A'.

The segment AA' is called the *transverse axis*, and BB' is called the *conjugate axis*.

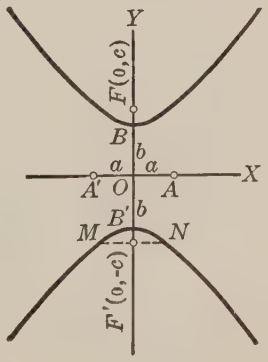
From the standard equation we see that when $y = 0$, we have $x = \pm a$; that is, a represents the transverse semiaxis.

When $x = 0$, however, $y = \pm b\sqrt{-1}$; that is, the curve does not cut the y axis. By finding the value of b from the relation $b^2 = c^2 - a^2$, we can lay off the conjugate axis BB' equal to $2\,b$.

The length of the chord MN, the perpendicular through a focus to the transverse axis, is called the *focal width* (w_f). As on page 213, it is easily shown that

$$w_f = \frac{2\,b^2}{a}.$$

Foci on the y Axis. If the foci are taken on the y axis at equal distances from the origin, as here shown, the transverse axis falls on the y axis and the conjugate axis on the x axis, thus interchanging the x and y terms; that is, the equation of this hyperbola is



$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

In this case $a^2 = c^2 - b^2$, the constant difference is $\pm 2\,b$, and for the focal width w_f we have

$$w_f = \frac{2\,a^2}{b}.$$

Asymptotes. In the figure below let $P'P$ be a line through the center O intersecting the hyperbola at P' and P. As P moves out on the curve, $P'P$ approaches either QR or ST.

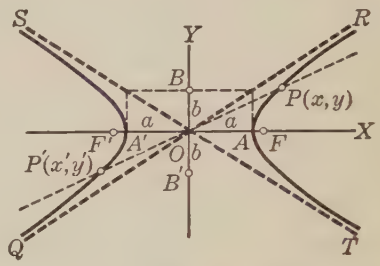
Using the intercept-slope formula and letting $P(x, y)$ be any point on $P'P$, we may write the equation of this line as

$$y = mx + 0.$$

Substituting mx for y in the standard equation, we have

$$\frac{x^2}{a^2} - \frac{m^2x^2}{b^2} = 1,$$

or $$m = \pm\sqrt{\frac{b^2}{a^2} - \frac{b^2}{x^2}}.$$



Now, as P moves out indefinitely far on the curve, the abscissa x of P increases without limit. Since b is a constant, $\frac{b^2}{x^2} \to 0$, and hence $m \to \pm\frac{b}{a}$. The equations of QR and ST, which are called the *asymptotes* of the hyperbola, are therefore

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x.$$

Exercises. The Hyperbola

In each of the following cases find the equation and then draw the hyperbola which is determined as follows:

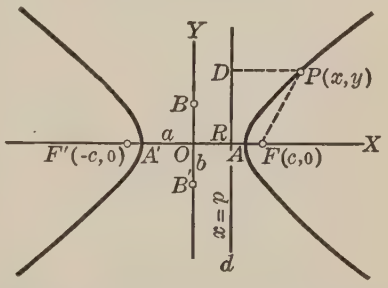
1. Foci, (5, 0) and (− 5, 0); transverse axis, 6.
2. Vertices, (0, 4) and (0, − 4); conjugate axis, 6.
3. Vertices, (5, 0) and (− 5, 0); conjugate axis, 10.
4. Vertices, (3, 0) and (− 3, 0); focal width, $2\frac{2}{3}$.
5. Asymptotes, $3x = 4y$ and $3x = -4y$; focus, $(\pm 5, 0)$.
6. Transverse axis, 12; conjugate axis, 8; foci on the x axis.
7. Transverse axis, 12; conjugate axis, 8; foci on the y axis.

Boscovich's Definition. As in the case of the ellipse (page 215), the hyperbola may be defined as the locus of a point which moves so that the ratio of its distance from a given point (the *focus*) to its distance from a given line (the *directrix*) is a constant, which in this case is greater than 1.

That is, representing this constant (the *eccentricity*) by e, the condition that $P(x, y)$ in the figure below generates a hyperbola is that

$$\frac{FP}{DP} = e > 1.$$

If, as before, we take a as the transverse semiaxis, $F(c, 0)$ as the focus, and $x = p$ as the directrix, we have



$$\frac{AF}{RA} = \frac{c-a}{a-p} = e,$$

and $$\frac{A'F}{A'R} = \frac{c+a}{a+p} = e;$$

or $$c - a = ae - ep,$$

and $$c + a = ae + ep.$$

Subtracting and adding these two equations, we then have

$$a = ep \text{ and } c = ae = e^2p.$$

Replacing b^2 by its equal, $c^2 - a^2$, in the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

we have

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1.$$

Hence in terms of the eccentricity e and the distance p of the directrix from the origin, the equation of the hyperbola is

$$\frac{x^2}{e^2p^2} - \frac{y^2}{e^2p^2(e^2-1)} = 1.$$

As with the ellipse, the hyperbola may be equally well defined from the focus $F'(-c, 0)$ and the directrix $x = -p$.

Exercises. The Hyperbola

1. From the relations $a^2 = e^2p^2$ and $b^2 = e^2p^2(e^2 - 1)$ derive a formula for e in terms of a and b and apply it in finding the eccentricity of the hyperbola $\frac{1}{16}x^2 - \frac{1}{4}y^2 = 1$.

2. In Ex. 1 find the equations of the directrices.

In each of the following cases find the transverse and conjugate axes, the coordinates of the foci and vertices, the eccentricity and focal width, and the equations of the directrices:

3. $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

4. $\frac{x^2}{25} - \frac{y^2}{4} = 1$.

5. $x^2 - 4y^2 = 4$.

6. $x^2 - y^2 = 9$.

7. $3x^2 - 8y^2 = 24$.

8. $13x^2 - 5y^2 = 65$.

9. $\frac{y^2}{16} - \frac{x^2}{4} = 1$.

10. $\frac{y^2}{25} - \frac{x^2}{9} = 1$.

11. $y^2 - x^2 = 1$.

12. $6y^2 - 4x^2 = 24$.

13. $18y^2 - 3x^2 = 54$.

14. $2y^2 - x^2 = 4$.

15. Sketch the hyperbolas which are represented by the equations $\frac{1}{9}x^2 - \frac{1}{6}y^2 = 1$ and $\frac{1}{10}x^2 - \frac{1}{5}y^2 = 1$ and show that they have the same foci.

In each of the following cases locate the vertices and the ends of the focal chord perpendicular to the transverse axis, draw the asymptotes, and sketch the hyperbola:

16. $x^2 - y^2 = 16$.

17. $6x^2 - 8y^2 = 24$.

18. $2y^2 - x^2 = 1$.

19. $4y^2 - 5x^2 = 20$.

Draw the hyperbolas which are determined as follows, and write the equation of the locus below each figure:

20. Vertices, $(5, 0)$ and $(-5, 0)$; foci, $(10, 0)$ and $(-10, 0)$.

21. Directrices, $x = 6$ and $x = -6$; eccentricity, $1\frac{1}{2}$.

22. Focal width, 6; eccentricity, $\sqrt{2}$; foci on the x axis.

8. Miscellaneous Loci

Vertical Asymptote. In plotting the locus of an equation by the ordinary graphic methods, we sometimes find that assigning a certain value to one of the variables gives a value for the other variable that cannot be plotted. For example, if we solve for y the equation

$$xy - x - 2y = 0,$$

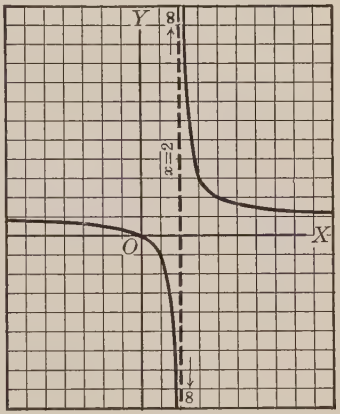
we have

$$y = \frac{x}{x-2}.$$

If we now let $x = 2$, we have $y = \frac{2}{0}$, and since division by zero is not permitted, we cannot locate a point for y which corresponds to the value 2 for x.

We may, however, let x approach 2 through values greater than 2, as in this table:

If $x =$	3	2.5	2.2	2.1	$\cdots$
$y =$	3	5	11	21	$\cdots$



We thus see that y is always positive and increases rapidly.

Similarly, if x is made to approach 2 through values less than 2, the table is as follows:

If $x =$	1	1.5	1.8	1.9	$\cdots$
$y =$	-1	-3	-9	-19	$\cdots$

We thus see that y is now negative and decreases rapidly.

We may therefore say that, as $x \to 2$, $y \to \pm\infty$, which means that y increases or decreases without limit as $x \to 2$.

Instead of using the expression "as $x \to 2$, $y \to \pm\infty$," this is sometimes written "when $x = 2$, $y = \pm\infty$."

That is, the curve approaches, but does not meet, the line $x = 2$, which is called a *vertical asymptote.*

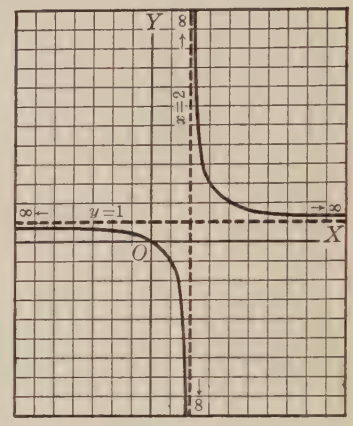
Horizontal Asymptote. In the case on page 223 we found that the line $x = 2$ is the vertical asymptote of the equation

$$xy - x - 2y = 0.$$

If we solve for x, we have

$$x = \frac{2y}{y-1},$$

from which we see that as $y \to 1$, $x \to \pm\infty$. The line $y = 1$ is therefore called a *horizontal asymptote* of the curve.



In general, therefore, to find vertical asymptotes we try to find a fractional value for y in terms of x, and then look for values of x that will make the denominator (but not the numerator) of this fraction zero. To find horizontal asymptotes we solve for x and proceed in the same way.

Thus, if the given equation is $x^2y - y - 2 = 0$, we have

$$y = \frac{2}{x^2-1} \quad \text{and} \quad x = \pm\sqrt{\frac{y+2}{y}}.$$

Hence the lines $x = 1$ and $x = -1$ are the vertical asymptotes and the line $y = 0$ is the horizontal asymptote, as the student should verify by drawing the curve and its asymptotes.

Exercises. Asymptotes

Find the vertical and horizontal asymptotes in each of the following cases and draw the figure:

1. $xy - y = 2$.
2. $xy^2 - x = 4$.
3. $y = \dfrac{x^2}{x^2-1}$.
4. $x = \dfrac{y^2}{y^2-4}$.
5. $2xy + 1 = 0$.
6. $y^2 - xy = -2$.
7. $y^2 = \dfrac{x^2}{2-x}$.
8. $x^2 = \dfrac{y^3}{2-y}$.
9. $x^2y - y = 8$.
10. $y^2 + xy = 6$.
11. $y = \dfrac{x+2}{x-3}$.
12. $x = \dfrac{y-2}{y+3}$.

The Cissoid. In the figure below let OA be the diameter of a circle with the radius r, let ST be tangent to the circle at A, and let OQ be any line from O to ST, cutting the circle at R. If we now let OQ revolve about O, we shall find that the locus of P, a point on the line such that $OP = RQ$, is the heavy curve shown. This locus is called a *cissoid*.

The name "cissoid" means "ivy-like."

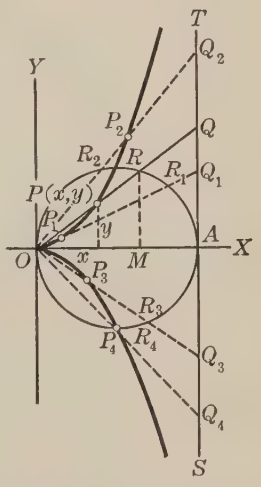
Any number of points, as P_1, P_2, P_3, P_4, $\cdots$, can be located by using ruler and compasses, and the curve can then be drawn through them.

Equation of the Cissoid. In finding the equation of the cissoid, we may take the diameter OA of the circle as the x axis, and O as the origin. Then, letting $P(x, y)$ be any point on the curve and drawing RM perpendicular to OX, we have

$$\frac{y}{x} = \frac{RM}{OM},$$

or

$$y = \frac{x \cdot RM}{OM}.$$



Then, since $RQ = OP$, we see that $AM = x$ and hence

$$OM = 2r - x.$$

Since RM is the mean proportional between AM and OM, we then have

$$RM = \sqrt{AM \cdot OM}$$
$$= \sqrt{x(2r - x)}.$$

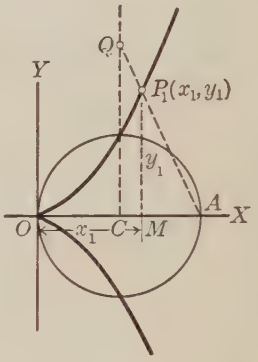
Substituting these values of OM and RM in the above equation for y and squaring, the result, which is the equation of the cissoid, is

$$y^2 = \frac{x^3}{2r - x}.$$

When $x \longrightarrow 2r$, we see from the equation of the cissoid that $y \longrightarrow \pm\infty$; that is, $x = 2r$ is a vertical asymptote.

Duplication of the Cube. The cissoid was invented by Diocles, a Greek mathematician of the second century B.C., who used it to find the edge of a cube having twice the volume of a given cube. This ancient problem, known as "the duplication of the cube," may be solved as follows:

At C, the center of the circle determining the cissoid, draw a perpendicular to OX, and on it lay off $CQ = 2r$. Then draw QA cutting the cissoid at $P_1(x_1, y_1)$, and draw P_1M perpendicular to OX. Now since



$$CA = r = \tfrac{1}{2}\, CQ,$$

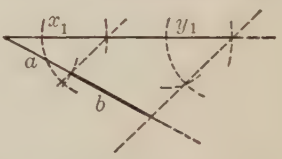
$$MA = 2r - x_1 = \tfrac{1}{2}\, MP_1 = \tfrac{1}{2}\, y_1.$$

Then, since P_1 is on the cissoid,

$$y_1^2 = \frac{x_1^3}{2r - x_1} = \frac{x_1^3}{\frac{1}{2} y_1};$$

whence $$y_1^3 = 2x_1^3.$$

Hence, if the given cube has the edge a, we need only construct with ruler and compasses a fourth proportional b to x_1, y_1, and a, as shown below; that is, we construct the segment b such that



$$\frac{x_1}{y_1} = \frac{a}{b};$$

whence $$b^3 = \frac{a^3 y_1^3}{x_1^3}.$$

Now, as shown above, $y_1^3 = 2x_1^3$, and hence $b^3 = 2a^3$; that is, we have thus found b, the edge of the cube which has twice the volume of the given cube.

By taking $CQ = nr$, in the first figure above, we can find the edge of a cube which has a volume n times that of a given cube.

Although, as shown in the second figure above, the fourth proportional b is easily constructed with only an unmarked ruler and the compasses (the only instruments permitted in a purely geometric construction), the cissoid itself cannot be so constructed.

The Conchoid. In the figure below let O be a fixed point and l a fixed line. If a line through O cutting the fixed line at Q revolves about O, the locus of a point P on the revolving line at a constant distance b from Q is the heavy curve shown in the figure. This locus is called a *conchoid.*

The point P may be taken to the left of l, as at P', thus tracing the second branch of the curve.

The name "conchoid," which means "shell-shaped," was given to the curve on account of the supposed resemblance to a mussel shell.

Points on the curve are easily located by drawing a number of lines through O, as suggested by the dotted lines, cutting the fixed line l at $Q_1, Q_2, Q_3, \cdots$. With these points as centers and b as radius, we may then lay off the points P_1, P_1'; P_2, P_2'; P_3, P_3'; and so on. (The point P_3' in the figure coincides with O.)

Equation of the Conchoid. To derive the equation of the conchoid we may take the fixed point O as the origin, and let the line $x = a$, which is a units to the right of O, be the given fixed line. Then, taking $P(x, y)$ as any point on the curve to the right of l, we have

$$\frac{OP}{QP} = \frac{OM}{RM},$$

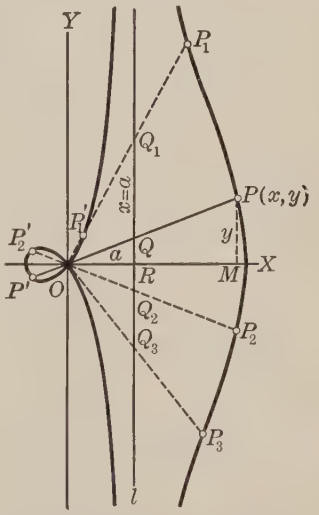
in which $OP = \sqrt{x^2 + y^2}$, $QP = b$, $OM = x$, and $RM = x - a$; that is,

$$\frac{\sqrt{x^2 + y^2}}{b} = \frac{x}{x - a};$$

whence $(x^2 + y^2)(x - a)^2 = b^2x^2$.

The equation of the conchoid may then be written as

$$y^2 = \frac{b^2x^2 - x^2(x - a)^2}{(x - a)^2}.$$

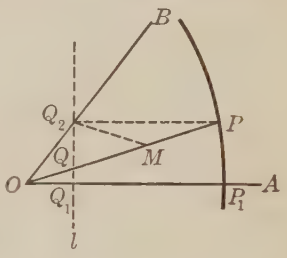


The same equation is easily derived by taking P to the left of l. From the equation it is easily seen that the line $x = a$ is a vertical asymptote of the curve.

Trisection of an Angle. The conchoid was invented by Nicomedes, a Greek mathematician of the second century B.C., who used it in solving the ancient problem of trisecting any angle.

Except for constructing the conchoid, which is done by the method described on page 227, this problem can be solved by methods of elementary geometry, with which the student is already familiar.

Suppose that $\angle AOB$ is the angle that we wish to trisect. At any convenient point Q_1 on the arm OA we may first construct a perpendicular l, cutting the arm OB at Q_2, and then on OA we may lay off the segment Q_1P_1 so that $Q_1P_1 = 2\,OQ_2$.



Drawing the conchoid in which O is the fixed point, l the fixed line, and Q_1P_1 (or $2\,OQ_2$) the constant length, we may then construct a perpendicular to l at Q_2, letting it cut the curve at P.

If we then draw a line from O through P, we shall have

$$\angle AOP = \tfrac{1}{3}\,\angle AOB.$$

To prove this, let Q be the point where OP cuts l and take M as the midpoint of QP. Drawing Q_2M, which is a median of the rt. $\triangle QPQ_2$, we then have

$$Q_2M = QM = MP,$$

and hence $\angle Q_2PM = \angle MQ_2P.$

Now $\angle Q_2MO = \angle Q_2PM + \angle MQ_2P,$

and hence $\angle Q_2MO = 2\,\angle Q_2PM.$

Since P and P_1 are points on the conchoid, we have

$$QP = Q_1P_1 = 2\,OQ_2,$$

and hence $Q_2M = OQ_2.$

Then $\angle POB = \angle Q_2MO = 2\,\angle Q_2PM = 2\,\angle AOP,$

or $\angle AOP = \tfrac{1}{3}\,\angle AOB.$

The Constant π. In elementary geometry the constant π is defined as the ratio of the circumference of a circle to its diameter; that is, in the circle with the circumference C and the diameter d,

$$\pi = \frac{C}{d}.$$

The circumference C may also be regarded as the limit of the perimeter of a circumscribed or inscribed regular polygon of n sides, each of which has the length s. That is, if we let the number of sides increase without limit, we may define π by the method of limits as

$$\pi = \lim_{n \to \infty} \frac{ns}{d}.$$

This is read "π is the limit of $\frac{ns}{d}$ as n increases without limit."

The value of π to six figures is 3.14159, a value that can be found by the method of limits if desired.

For example, if we take a regular polygon inscribed in a circle with the diameter 1, we can find that, when $n = 12$, $ns = 3.10583$; when $n = 24$, $ns = 3.13263$; when $n = 48$, $ns = 3.13935$; when $n = 96$, $ns = 3.14103$, and so on until, when $n = 768$, $ns = 3.14159$. Since $d = 1$ in this case, we thus have the approximate value 3.14159 for π.

In practice, however, the value of π is obtained by finding the sum of one of several kinds of series developed by the methods of higher analysis, instead of using the above method.

The Constant e. Another important constant which occurs frequently in the application of mathematics to science is designated by the letter e. By the method of limits this constant may be defined as

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

The value of the constant e to four figures is 2.718.

The constant e is the base of a system of logarithms that is known as the hyperbolic system, the Napierian system, or the natural system. These logarithms are frequently employed in advanced mathematics.

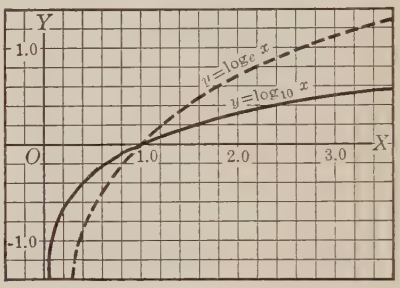
Logarithmic Curve. If we let y represent the logarithm of a number x to the base b, we have (page 65)

$$x = b^y,$$

or, when expressed in logarithmic notation, we have

$$y = \log_b x.$$

The locus of this equation is called a *logarithmic curve*, and to draw the curve when $b = 10$ we may take convenient values of x and find the corresponding values of y from a table of common logarithms. From these values we may plot the solid curve here shown.



When e is taken as the base, we have $y = \log_e x$. The locus of this equation, the dotted curve in the figure, may be plotted in the same way by using a table of natural logarithms.

If we take the logarithm to the base 10 of each side of the equation

$$x = e^y,$$

we have $\log_{10} x = y \log_{10} e.$

Then, since $\log_{10} e = \log_{10} 2.718 = 0.4343,$

we have $0.4343\, y = \log_{10} x,$

or $y = 2.3026 \log_{10} x.$

This is a convenient relation for converting common logarithms (those with the base 10) to natural logarithms (those with the base e).

In $x = b^y$ it is impossible for x to be negative for real values of y. Hence a negative number has no real logarithm, and the curve lies wholly to the right of the y axis.

Furthermore, as $x \to 0$, y must be negative and must increase numerically without limit; that is, the y axis is a vertical asymptote of the locus of $y = \log_b x$.

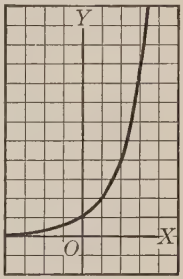
Exponential Curve. The locus of an equation of the form

$$y = ka^x,$$

where a is a positive constant, is called an *exponential curve.*

When $x = 0$ in the equation, we have $y = k$; that is, the curve cuts the y axis at a point k units above the origin.

Furthermore, when $a > 1$, it is evident that $y \to 0$ as $x \to -\infty$; that is, the x axis is a horizontal asymptote.



The figure at the right shows the locus of the equation $y = ka^x$ when $a = 2$ and $k = 1$; that is, of the equation $y = 2^x$. On account of the small scale to which the graph is drawn it is not possible to show the actual separation that exists between the curve and the x axis which it approaches as an asymptote; that is, when $x = -4$, $y = \frac{1}{16}$; when $x = -5$, $y = \frac{1}{32}$; and so on for other negative values of x.

In a case like $y = 2.25^x$, points may be easily located by writing the equation in the form

$$\log_{10} y = x \log_{10} 2.25 = 0.3522\, x.$$

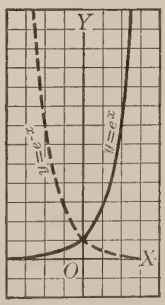
Then when $x = 1.5$, $\log_{10} y = 0.5283$, and hence $y = 3.38$, and so on.

Graphs of $y = e^x$ and $y = e^{-x}$. It is evident that the exponential equation $y = e^x$, where e is the base of the system of natural logarithms, has the same form as the above equation.

In drawing the graph of the equation $y = e^x$, we may locate the points either by using a table of natural logarithms or by writing the equation, as suggested on page 230, in the form

$$\log_{10} y = 0.4343\, x,$$

and then using common logarithms as described above for the equation $y = 2.25^x$.



The graph of the equation $y = e^x$ is shown by the solid curve in the figure, and the graph of $y = e^{-x}$ is shown by the dotted curve.

The graph of the equation $y = ke^x$ may be constructed from the graph of $y = e^x$ by multiplying each ordinate by k.

The Equation $y = ke^{mx}$. Equations of the form $y = ke^{mx}$, where m and k are constants, represent a type of exponential curves which have important applications in science. For example, the relation between the height h above sea level, measured in miles, and the atmospheric pressure P, measured by the length in inches of a column of mercury, may be expressed by means of the equation

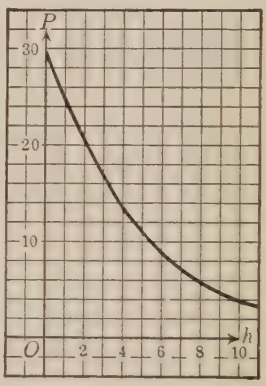
$$P = 29.92\, e^{-\frac{h}{5}}.$$

To draw the graph of this equation, we may construct the following table and then plot the curve shown below:

If $h =$	0	1	2	3	4	5	6	7	···
$P =$	29.9	24.5	20.1	16.4	13.4	11.0	9.0	7.4	···

If natural logarithms are not available, we may compute the values of P by taking

$$\log_{10} P = \log_{10} 29.92 - \tfrac{1}{5}\, h \log_{10} e$$
$$= 1.4760 - 0.0869\, h.$$



From this graph we can determine approximate values of P or h. For example, when the column of mercury is 22.5 in. long, the height (h) at which this is the atmospheric pressure is seen to be approximately 1.4 mi.

The value of h might have been computed directly from the equation itself; that is, when $P = 22.5$, we have

$$22.5 = 29.92\, e^{-\frac{h}{5}},$$

or, by taking the common logarithms of both members, we have

$$\log_{10} 22.5 = \log_{10} 29.92 - \tfrac{1}{5}\, h \log_{10} e.$$

Hence $$h = \frac{5(\log_{10} 29.92 - \log_{10} 22.5)}{\log_{10} e},$$

or $$h = \frac{5(1.4760 - 1.3522)}{0.4343} = 1.43, \text{ approximately.}$$

Exercises. Exponential Equations

Plot the loci of the following equations:

1. $y = \log_{10} 2x$.
2. $y = 3 \log_{10} x$.
3. $y = \log_{10} \sqrt{x}$.
4. $y = 3 \log_{10} \sqrt{x}$.
5. $y = 2 \log_e x$.
6. $y = 3^x$.
7. $y = e^{2x}$.
8. $y = e^{-x}$.
9. $y = 2e^{3x}$.
10. $y = e^{\frac{x}{2}}$.
11. $y = 2e^{-\frac{x}{4}}$.
12. $y = 2e^{-x^2}$.
13. $y = 3e^{-\frac{x^2}{4}}$.
14. $y = e^{-2x^2}$.
15. $y = \frac{1}{2} e^{-x}$.

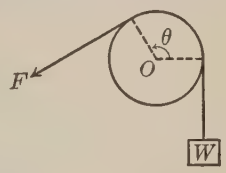
16. If $y = 3e^{-0.2x}$, what is the value of x when $y = 5$?

17. Plot the locus of the equation $y = 2 \log_{10}(2 + x)$.

18. Without plotting, show that the graph of the equation $y = e^{-x^2}$ is symmetric with respect to the y axis, and that the x axis is a horizontal asymptote. Then plot the graph and thus verify your conclusions.

Curves of this type are applied in the study of statistics.

19. When a weight W is suspended over a pulley, as here shown, the force F required to keep the weight from falling is given by the formula $F = We^{-a\theta}$, where a is a constant and θ is the angle shown in the figure. Find the number of degrees in the angle θ when $W = 500$, $F = 265$, and $a = 0.0053$.



20. Under certain conditions the equation $L = 100e^{-0.3t}$ expresses the relation between the amount of light L transmitted and the thickness t of the glass through which it passes. Find the value of L when $t = 2$; of t when $L = 95$.

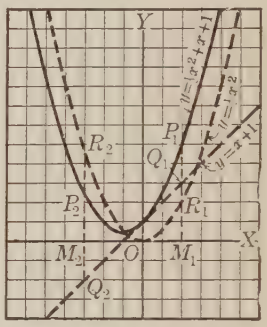
21. The law of increase for a certain kind of bacteria is $n = ke^{ht}$, where k is the initial number of bacteria, h is a constant, and n is the number at the end of t minutes. If 2000 bacteria double themselves in 30 min., what is the value of h? How many bacteria will there be at the end of 3 hr.?

Addition of Ordinates. By using a method known as the *addition of ordinates*, we can frequently build up the graph of an equation from the graphs of two or more others.

For example, to plot the locus of the equation

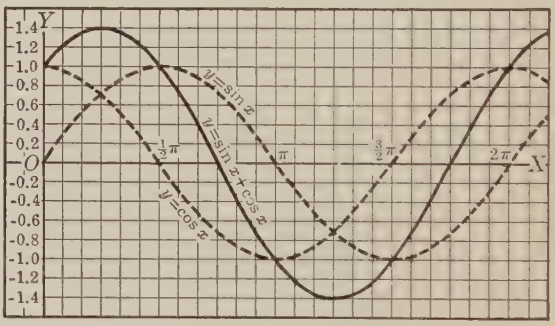
$$y = \tfrac{1}{2}x^2 + x + 1,$$

we may first construct the locus of the parabola $y = \frac{1}{2}x^2$ and then the locus of the line $y = x + 1$, as shown in the figure below. Points on the graph of $y = \frac{1}{2}x^2 + x + 1$ are then located by adding the ordinates of corresponding points on the parabola and the line.



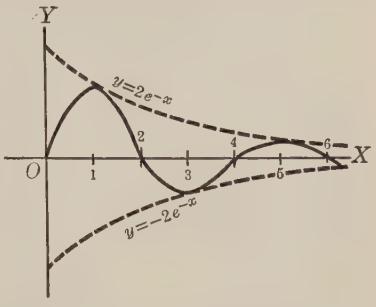
For any given value of x in the equation $y = \frac{1}{2}x^2 + (x + 1)$ the corresponding value of y is the algebraic sum of the values of y in $y = \frac{1}{2}x^2$ and $y = x + 1$. We may therefore lay off the segment M_1P_1 in such a way that $M_1P_1 = M_1R_1 + M_1Q_1$, and similarly we lay off the segment M_2P_2 in such a way that $M_2P_2 = M_2R_2 - Q_2M_2$.

Graph of $y = \sin x + \cos x$. This figure shows how the graph of the equation $y = \sin x + \cos x$ may be built up by the above method from the sine curve and the cosine curve on page 110.



It is often convenient in measuring angles to use what is known as *radian measure*. A *radian* is the central angle of an arc equal in length to the radius of the circle. Since the circumference is $2\pi r$, there are evidently 2π radians in the space about a point, or π radians in 180°, and 1 radian is equal to approximately 57.3°. In the above graph every space on the x axis represents $\frac{1}{12}\pi$, or 15°.

Graph of $y = 2\,e^{-x}\sin\frac{1}{2}\pi x$. In the case of the equation $y = 2\,e^{-x}\sin\frac{1}{2}\pi x$ we see that for odd integral values of x we have $y = \pm 2\,e^{-x}$, while for even integral values of x we have $y = 0$. Furthermore, since $\sin\frac{1}{2}\pi x \leqq 1$, we see that $2\,e^{-x}\sin\frac{1}{2}\pi x$ is never greater than $2\,e^{-x}$, and we thus see that the locus of $y = 2\,e^{-x}\sin\frac{1}{2}\pi x$ must lie between the loci of $y = 2\,e^{-x}$ and $y = -2\,e^{-x}$, which are of the form shown on page 231. We may then sketch the locus of $y = 2\,e^{-x}\sin\frac{1}{2}\pi x$ by using these two curves as shown above.



The figure is not drawn to scale, the shape of the curves $y = \pm 2\,e^{-x}$ being only roughly indicated in order to make a clear figure. As an idea of the degree of approximation, we see that when $x = 0$, the ordinates of $y = \pm 2\,e^{-x}$ are ± 2, while when $x = 6$, they are ± 0.005.

Exercises. Review of Miscellaneous Loci

Draw the loci of the following equations:

1. $y = x^2 + x - 3$.
2. $y = 4\,x^2 - 2\,x$.
3. $y = x + e^x$.
4. $y = 2\,x + 1 + e^x$.
5. $y = x + \log x$.
6. $y = e^x + e^{-x}$.
7. $y = \frac{1}{2}(e^x - e^{-x})$.
8. $y = x + \sin x$.
9. $y = x + \cos x$.
10. $y = \sin x + \sin 2\,x$.
11. $y = \sin 2\,x + \cos 2\,x$.
12. $y = 2\sin x + 3\cos x$.
13. $y = x\sin\pi x$.
14. $y = \sin\frac{1}{2}\,x + 3\cos\frac{1}{2}\,x$.
15. $y = e^x - \sin x$.
16. $y = e^{-x}\sin\pi x$.
17. $y = 2\,e^{-x}\cos\frac{1}{2}\pi x$.
18. $y = e^{-x}\sin 2\pi x$.
19. $y = \sin x + \cos x + \sin 2\,x$.
20. $y = \frac{1}{2}\sin\frac{1}{2}\,x + \frac{1}{2}\cos 2\,x$.

Intersections of Curves. As we have already seen, if a point lies on a locus, the coordinates of the point satisfy the equation of the locus. Hence if two loci intersect in one or more points, the coordinates of any point of intersection satisfy the equations of both loci.

Thus, if we are given two simultaneous equations in x and y, we may treat them from a geometric standpoint by drawing their graphs and finding the coordinates of the point of intersection. It frequently happens that this method gives only an approximate solution, and it is important to observe that, since the coordinates of a point are real numbers, only real solutions of the equations can be obtained. The method, however, is often valuable in showing the nature of the solution of the equations, and it reveals whether or not any pair of given equations have any real values of x and y in common.

The graphic solution of two equations was suggested in certain exercises on page 164, but is here reviewed in preparation for more difficult cases.

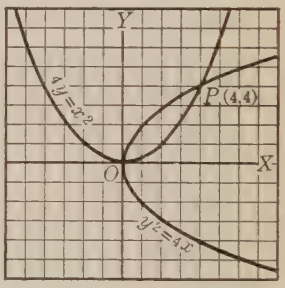
For example, if the given equations are

$$y^2 = 4x \qquad (1)$$

and

$$4y = x^2, \qquad (2)$$

we may draw their graphs, as here shown, and find that they meet at $O(0, 0)$ and $P(4, 4)$. That is, the values which satisfy the equations are $x = 0$, $y = 0$ and $x = 4$, $y = 4$.



In solving these two equations for real roots by the methods of pure algebra, we should first find from equation (2) that

$$y = \tfrac{1}{4}x^2.$$

Then substituting this value of y in equation (1), we have

$$\tfrac{1}{16}x^4 = 4x,$$

from which $x = 0$ or 4,

and $y = 0$ or 4.

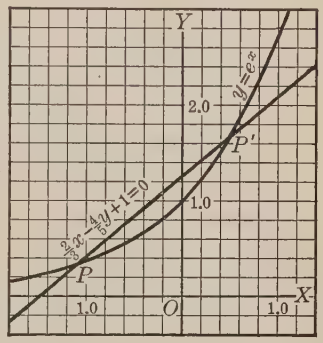
Graphic Solutions. The method described on page 236 can also be applied to a pair of equations or to a single equation which cannot be solved by the ordinary methods used in elementary algebra. The following examples illustrate the method of handling cases of this kind.

1. Find by means of graphs an approximate solution of the pair of equations

$$\tfrac{2}{3}x - \tfrac{4}{5}y + 1 = 0 \qquad (1)$$

and

$$y = e^x. \qquad (2)$$



Drawing the graphs of these two equations, as here shown, we find that they intersect at the points $P(-1.11,\ 0.33)$ and $P'(0.52,\ 1.68)$; that is, these values are reasonable estimates of the coordinates.

The closeness of the approximation which we obtain by the graphic method described above can be tested by substituting in the given equations the pairs of values $x = -1.11$, $y = 0.33$, and $x = 0.52$, $y = 1.68$.

By using paper ruled in millimeter squares and taking a larger unit it is often possible to make a closer approximation.

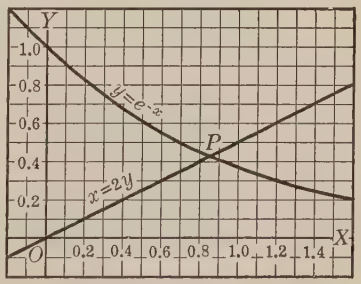
2. Find by graphs an approximate value of x satisfying the equation $x = 2\,e^{-x}$.

If we substitute y for e^{-x}, we have the two equations

$$y = e^{-x} \qquad (1)$$

and

$$x = 2\,y. \qquad (2)$$



Drawing the graphs of these two equations, as here shown, we find that the abscissa of P is approximately 0.85; that is, in the equation $x = 2\,e^{-x}$ we have $x = 0.85$.

In checking this estimate by subtituting 0.85 for x in the given equation, we may write $\log_{10} x = \log_{10} 2 - x \log_{10} e$, and then use a table of common logarithms.

Exercises. Solution of Equations

By solving each of the following pairs of equations find the points of intersection of their loci, and check the results by drawing the graphs:

1. $y^2 = x$
   $x = y + 2$

2. $x^2 + y^2 = 25$
   $x + y - 1 = 0$

3. $xy = -5$
   $y = x + 6$

4. $x^2 - y^2 = 48$
   $11x - 13y = 0$

5. $x - y = 4$
   $x^2 - xy = 24$

6. $xy = 4$
   $x^2 + y^2 = 17$

7. $x^2 + y^2 = 36$
   $y^2 = 16x$

8. $9x^2 + 16y^2 = 144$
   $y^2 = \frac{9}{2}x$

Draw the graphs of the following pairs of equations, estimate the coordinates of the points of intersection, and thus obtain approximate solutions:

9. $y = 2x^2 + x - 3$
   $y = 3x + 5$

10. $y^2 - 2x = 4$
    $2x^2 + y = 3$

11. $\frac{x^2}{16} + \frac{y^2}{9} = 1$
    $\frac{x^2}{9} - \frac{y^2}{4} = \frac{16}{9}$

12. $y = e^x \sin \pi x$
    $y = x - 1$

13. $y = x^3$
    $2x + 3y = 6$

14. $y = \frac{1}{2}e^x$
    $y = x + 2$

15. $y^2 = \frac{x^3}{4 - x}$
    $\frac{y^2}{4} = x$

16. $y = \sin x$
    $2y + 3x = 1$

In Exs. 9–16 the student should give at least a rough check by substituting the values of x and y in the equations.

Solve the following equations by means of graphs:

17. $x = 0.1\, e^{2x}$.

18. $x = 0.2\, e^{\frac{x}{2}}$.

19. $x + 1 = \sin x$.

20. $2x + 3 = \cos x$.

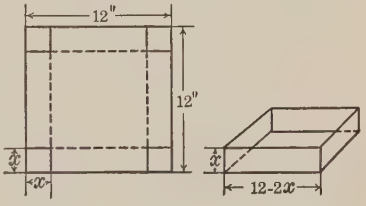
CHAPTER VI

THE CALCULUS

1. Maximum and Minimum Values

Maximum and Minimum Values. In practical problems we often need to know the greatest or the least value, called the *maximum* or *minimum value* respectively, that an expression can have within a certain range.

A Problem Involving a Maximum. To take a simple case, suppose that an open box is to be made from a square piece of tin 12 in. on a side by cutting squares from each corner and turning up the flaps, as here shown. What length of side must the corner squares have in order to give the box the maximum volume?

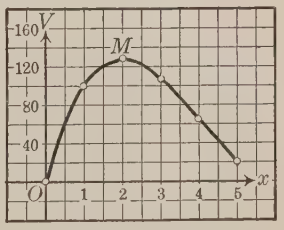


If we let x represent the side of the small squares, the side of the bottom of the box will then be $12 - 2x$, and the volume V will be represented by the equation

$$V = (12 - 2x)^2 \cdot x = 4x^3 - 48x^2 + 144x.$$

Assigning convenient values to x, we may then make the following table of values:

If $x =$	0	1	2	3	4	5	···
$V =$	0	100	128	108	64	20	···



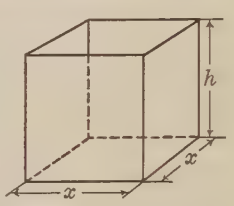
Sketching the graph, as here shown, it appears that the greatest value of V occurs at M, where $x = 2$. That is, the box will have a maximum volume if the squares cut from each corner are 2 in. on a side.

A Problem Involving a Minimum. As an illustration of a case in which a minimum is required, suppose that a manufacturer has an order for a quantity of open metal containers which are to have square bases and a capacity of 10 cu. ft. If the base is to be made of heavy sheet metal costing 10¢/sq. ft., and the sides are to be made of metal costing 5¢/sq. ft., what dimensions will give a minimum cost for materials?

If we let x represent the length of the side of the base and h the height, both dimensions being expressed in feet, we have

$$x^2h = 10,$$

from which $$h = \frac{10}{x^2}.$$



The number of square feet of metal needed for the base is evidently x^2 and that needed for the four sides is $4\,xh$.

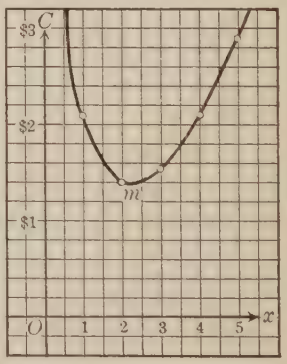
Letting C represent the total cost in cents of the material needed for both the bottom and the sides, we then have

$$C = 10\,x^2 + 5 \cdot 4\,xh,$$

or $$C = 10\,x^2 + \frac{200}{x}.$$

In this equation we wish to find the value of x that will make C a minimum. By assigning convenient values to x and computing C to the nearest 1¢, we have the following table:

$x =$	0	1	2	3	4	5	···
$C =$	∞	210	140	157	210	290	···



In sketching the graph here shown we find that we need additional points in the vicinity of $x = 2$. For example, we may take $x = 2.1$ and $x = 2.2$, for each of which we find that $C = 139$, to the nearest 1¢.

That is, the minimum m for C probably occurs between $x = 2.1$ and $x = 2.2$, and is approximately \$1.39, as is also evident from the graph. Thus, by making the base about 2.15 ft., or 2 ft. 1.8 in. on a side, and by making the height about 2 ft. 1.9 in., the cost of the material needed will be the least.

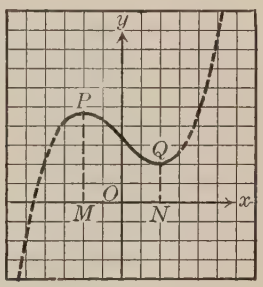
Maximum and Minimum Ordinates. In the problem on page 239 we were interested only in the maximum value of V, while in the one on page 240 we were interested in the minimum value of C. We may, however, wish to find both a maximum and a minimum value in the same problem. For example, taking a range of values from $x = -3$ to $x = 3$, let us consider the graph of the equation

$$y = \tfrac{1}{12}x^3 - x + \tfrac{10}{3}.$$

As usual, we first prepare a table of the corresponding values of x and y in the equation, as follows:

If $x =$	-3	-2	-1	0	1	2	3
$y =$	4.1	4.7	4.3	3.3	2.4	2	2.6

We then sketch the graph, as shown in the figure at the right.



From the figure it appears that the ordinate MP is greater than the ordinates just to the right or left of MP, and hence MP is said to be a *maximum ordinate.* Similarly, NQ appears to be less than the ordinates just to the right or left of Q, and is called a *minimum ordinate.*

It should also be observed that the terms "maximum ordinate" and "minimum ordinate" are purely relative. If we proceed farther to the right on the curve, as shown by the dotted lines, we shall find ordinates greater than MP; for example, when $x = 5$, we have $y = 8\frac{3}{4}$. That is, in describing MP as a maximum ordinate we mean simply that it is greater than any ordinate in the immediate vicinity of MP, and similarly in describing a minimum ordinate.

It is evident that the graphic method of determining maximum and minimum ordinates is at best only approximate. In the calculus which we shall soon take up we shall learn methods of finding these ordinates with precision.

Exercises. Maxima and Minima

Draw the graphs of the following equations and find the approximate maximum and minimum ordinates:

1. $y = 2x^2 + 3x + 5$.
2. $y = -5x^2 + 6x + 8$.
3. $y = x^2 - 3x + 4$.
4. $y = x^3 - 20x$.
5. $y = x^3 + 4x^2 + 8$.
6. $y = \frac{1}{3}x^3 - x^2 + 4$.
7. $y = 2x^3 - 5x^2 + 2$.
8. $y = -x^3 + 2x^2 - 1$.
9. $y = x^3 - 8x^2 + 13x + 4$.
10. $y = \frac{1}{3}x^3 - 2x^2 + 3x + 2$.

11. In the equation $P = 2c - 0.01c^2$ determine the approximate value of c which makes P a maximum, and find this value of P.

12. If a body is projected upward with an initial velocity of v_0 feet per second, the distance d, in feet, to which it will rise is given by the formula $d = v_0 t - \frac{1}{2}gt^2$, where g, the acceleration of gravity, is approximately 32 ft./sec. Find the time in which a baseball thrown vertically upward with an initial velocity of 96 ft./sec. will reach its highest point, and find the height which it reaches.

13. Solve Ex. 12 for a rifle bullet fired vertically upward with an initial velocity of 2000 ft./sec.

14. In the study of electricity it is shown that the amount of current C produced by a certain arrangement of n cells is given by the formula $C = \dfrac{1.9n}{0.01n^2 + 0.25}$. Draw the graph of this formula for values of n from 4 to 10, and determine what number of cells produces the largest current.

15. A rectangular flower garden is to have a walk 3 ft. wide on each long side and a walk 4 ft. wide across each end. What dimensions will make the total area of the walks and garden a minimum, and still leave an area of 432 sq. ft. for the garden itself?

2. Variables

Function. Before going further in our study of graphs, it will be a good plan to review some of the technical terms that we need to use in discussing how two or more quantities vary with respect to each other. For example, on page 34 we learned that one quantity is said to be a *function* of a second quantity when it depends upon the second in such a way that the first is determined when the second is specified. Thus, from the falling-body formula

$$d = \tfrac{1}{2}\, gt^2,$$

where g is the acceleration due to gravity with an approximate value of 32 ft./sec., the distance d traveled by the body can be determined for any given interval of time t; that is, d is a function of t.

As we have also learned, the function symbol $f(\ )$ is often a convenient abbreviation. For example, since $2x^2 - x + 1$ is a function of x, we may write

$$f(x) = 2x^2 - x + 1,$$

and then in the discussion refer to $f(x)$. If we wish to speak of the value of the function when $x = 2$, we may write $f(2) = 7$.

Variables. In the formula $d = \frac{1}{2}\, gt^2$, mentioned above, g is normally a constant, and hence the variables are d and t. Since we usually think of d as depending on t for its value, we call d the *dependent variable* and t the *independent variable.*

That is, if we are finding the distances that the body falls in 2 sec., we substitute the value 2 for t, and thus determine the dependent variable d.

We might, of course, wish to find the time it takes a body to fall a certain specified distance. In this case we should solve for t and have

$$t = \sqrt{\frac{2\,d}{g}}.$$

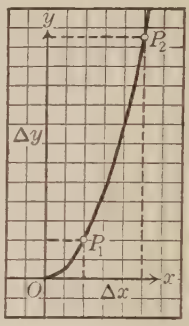
When written in this form, t becomes the dependent variable and d the independent variable.

In the expression $f(x) = 2x^2 - x + 1$, $f(x)$ is considered to be the dependent variable.

Changes in the Variables. We often wish to examine the changes produced in the dependent variable by certain specified changes in the independent variable. For example, in the case of

$$y = \tfrac{1}{2}x^2,$$

between the range of values from $x = 2$ to $x = 5$, we see that when $x = 2$ we have $y = 2$, and when $x = 5$ we have $y = 12.5$. That is, as shown by the accompanying graph, a change from 2 to 5 in the independent variable x produces a change of 10.5 in the dependent variable y.

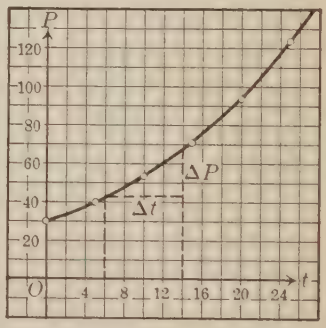


It is customary to use the symbol Δx (read "delta x") to designate the specified change in x and the symbol Δy to designate the corresponding change in y.

For the interval from $x = 2$ to $x = 5$, above, $\Delta x = 3$ and $\Delta y = 10.5$.

Calculation of Changes from a Graph. The graph of a functional relation furnishes an approximate method of calculating Δy for a given Δx, particularly when the functional relation is expressed by a table of values. For example, let us take the following table, which shows the vapor pressure P, measured in millimeters on a column of mercury, of methyl alcohol at various temperatures t (centigrade):

$t =$	0	5	10	15	20	25
$P =$	30	40	54	71	94	123



Drawing the graph, as here shown, we see that in considering the changes from $t = 6°$ to $t = 14°$ we have $\Delta t = 8$, and from the graph we find that $\Delta P = 25$, approximately.

Exercises. Functional Relations

1. When $f(x) = x^2 - 5x + 6$, find $f(-3)$, $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

2. Using the values found in Ex. 1, draw the graph of $f(x) = x^2 - 5x + 6$.

3. When $f(t) = 140t - 16t^2$, find $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$, and draw the graph of the given function.

4. Express the volume of a cube as a function of the edge e.

Find Δx and Δy in each of the following functional relations for the interval specified:

5. $y^2 = x^2 - 5x + 6$, between $x = 3$ and $x = 5$.

6. $y = x^3 + 1$, between $x = 1$ and $x = 4$.

7. $y = \sin x$, between $x = 60°$ and $x = 90°$.

8. Draw the graph of the falling-body formula $d = \frac{1}{2}gt^2$, taking the approximate value 32 for g, and find Δt and Δd in the interval between $t = 2$ and $t = 6$.

9. In the formula $P = 2c - 0.01c^2$ find ΔP for the interval between $c = 1$ and $c = 3$.

10. The formula $d = v_0 t - \frac{1}{2}gt^2$ gives the distance d, in feet, to which a body projected vertically upward at v_0 feet per second will rise in t seconds. Taking $v_0 = 140$ ft./sec. and $g = 32$, find Δd for the interval between $t = 2$ and $t = 4$.

11. The relation between the quantity of water Q, in cubic feet per minute, discharged from a 10-inch drain and the grade s of the pipe, expressed as a per cent, is as follows:

$s =$	1	2	3	4	5	···	10
$Q =$	116	146	170	190	208	···	279

Draw the graph and find ΔQ when $\Delta s = 1.5$ in the interval from $s = 3\%$ to $s = 4.5\%$.

Average Rate of Change. In addition to considering the amount of change in the function (dependent variable) for a given change in the variable, it is important to consider the average rate of change in the function. For example, in the functional relation

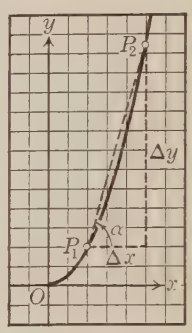
$$y = \tfrac{1}{2}x^2,$$

considered on page 244, we found that for the interval from $x = 2$ to $x = 5$ we have

$$\Delta x = 3$$

and

$$\Delta y = 10.5.$$



In finding the average rate at which Δy is increasing with respect to the interval Δx we therefore have

$$\frac{\Delta y}{\Delta x} = \frac{10.5}{3} = 3.5.$$

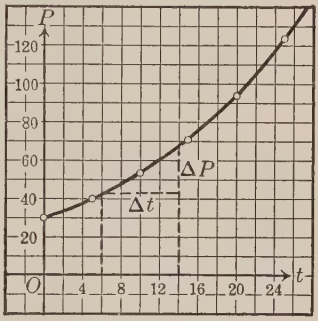
That is, each change of 1 in the variable x produces a change of 3.5 in y. In general, therefore, we see that

The average rate of change of a function with respect to the variable is found by dividing the total change in the function by the total change in the variable for the specified interval.

It is important to observe that the average rate of change depends upon the interval chosen. For example, for the interval from $x = 0$ to $x = 2$ in the above case we have $\Delta x = 2$ and $\Delta y = 2$, and hence for the average rate of change we have $\frac{\Delta y}{\Delta x} = \frac{2}{2} = 1$.

As we learned on page 174, $\frac{\Delta y}{\Delta x}$ is the slope of a line in terms of the difference between the abscissas and the difference between the ordinates of two points on the line. Hence, as is also seen from the graph, the average rate of change for the interval Δx is the slope of the line joining the points (P_1 and P_2) which correspond respectively to the initial and the final values of x in the interval Δx.

Average Rate in a Graph. Although the results will be only approximate, we can also determine the average rate of change in the case given at the foot of page 244, where the functional relation between the vapor pressure P of methyl alcohol and the temperature t was represented by this graph.



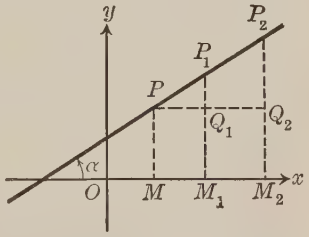
Taking the approximate value of ΔP, as determined from the graph, for the interval from $t = 6°$ to $t = 14°$, we have

$$\frac{\Delta P}{\Delta t} = \frac{25}{8} = 3.125.$$

That is, the average change in pressure per degree of temperature between 6° C. and 14° C. is 3.125 mm.

Uniform Rate of Change. A rate of change in the function with respect to the variable which is the same for all intervals is called a *uniform rate.*

If $\frac{\Delta y}{\Delta x}$ is uniform, we see from this figure that, for any two intervals MM_1 and MM_2,



$$\frac{Q_1P_1}{PQ_1} = \frac{Q_2P_2}{PQ_2};$$

that is, the slope of the line joining the points P and P_1 on the graph is the same as that of the line joining P and P_2.

Hence P, P_1, and P_2 lie in a straight line, or, in general,

If the average rate of change of a function with respect to the variable is uniform, the graph of the function is a straight line.

Conversely, since the slope of a straight line is constant,

If the graph of a function is a straight line, the rate of change of the function with respect to the variable is uniform.

Speed. In a functional relation between the time t and the distance d passed over by a moving body, the rate of change of d with respect to t gives the change of distance per unit of time, which is the *speed*, or *velocity*, of the body.

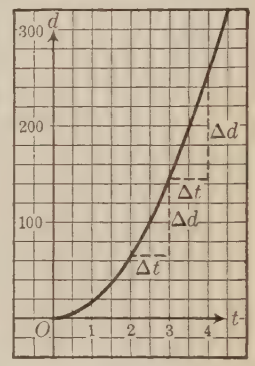
For example, if we take the value of g as 32 in the falling-body formula

$$d = \tfrac{1}{2}\, gt^2,$$

we have $d = 64$ when $t = 2$, and $d = 144$ when $t = 3$; that is, $\Delta d = 80$ for $\Delta t = 1$. Hence for the average speed v of the body in this interval we have

$$v = \frac{\Delta d}{\Delta t} = \frac{80}{1} = 80;$$

that is, the average speed is 80 ft./sec. during the third second of fall.



In the same way, for the fourth second of fall we have

$$v = \frac{\Delta d}{\Delta t} = \frac{112}{1} = 112;$$

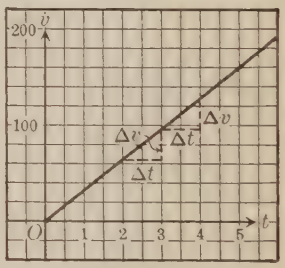
that is, the average speed in the next second is 112 ft./sec.

Acceleration. The rate of change of speed with respect to time is called the *acceleration*.

In the study of physics it is shown that the speed v of a falling body is given by the formula

$$v = gt.$$

Since the acceleration, or the rate of change of v with respect to t, is constant, the graph of this formula is a straight line, as here shown. In such a case the acceleration is said to be *uniform*.



In drawing the graph, we have taken the approximate value of 32 ft./sec. for g instead of the normal value of 32.16 ft./sec.

Exercises. Rate of Change

Draw the graph of each of the following equations and find the average rate of change of the dependent variable y with respect to x for the interval specified:

1. $y = \frac{1}{8} x^3$, between $x = 1$ and $x = 4$.
2. $y = x^2 - 5x + 2$, between $x = \frac{1}{2}$ and $x = 3$.
3. $y = x^2 + x$, between $x = 0$ and $x = 2$.
4. $y = x^3 + 2$, between $x = 1$ and $x = 4$.
5. $y = \frac{1}{x}$, between $x = 1$ and $x = 3$.
6. $y = x^2 + \frac{1}{x}$, between $x = 2$ and $x = 4$.

7. Taking the value of g as 32.16 in the falling-body formula $d = \frac{1}{2} gt^2$, calculate the average rate of change of d with respect to t for each of the following intervals: $t = 1$ to $t = 2$; $t = 1$ to $t = 1.5$; $t = 1$ to $t = 1.1$; $t = 1$ to $t = 1.01$.

8. If the relation between the speed v of a body, in feet per second, and the time t, in seconds, is $v = 2t^2 + 0.5t$, what is the average acceleration during the third second?

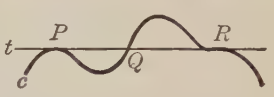
9. Taking the equation $V = \frac{450}{P}$ as representing the relation between the volume V, in cubic feet, and the pressure P, in pounds per square inch, find the average rate of change of V with respect to P in the interval from $P = 20$ to $P = 25$. What does the negative value of $\frac{\Delta V}{\Delta P}$ indicate?

10. The following table shows the population P, in millions, of the United States for a certain period of years (t):

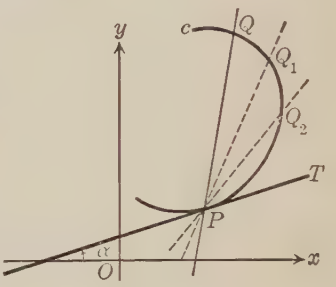
$t =$	1830	1840	1850	1860	1870	1880	1890	1900	1910	1920
$P =$	12.9	17.1	23.2	31.4	38.6	50.2	62.9	76.0	92.0	105.7

By means of a graph determine the average rate of change in the population during each 10-year period.

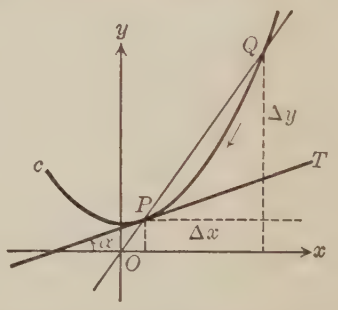
Tangent to a Curve. In this figure the line t appears to be tangent to the curve c at P, although it cuts the curve again at Q, and it also appears to be tangent to c at R, although it cuts c at R. The familiar definition of a tangent to a circle as the line having one and only one point in common with the circle does not, therefore, apply to all curves.



Instead, we may take any two points on the curve, as P and Q on the curve c in this figure, and let Q move along c (through the positions Q_1, Q_2, $\cdots$), approaching P as its limiting position. The tangent to the curve at P is then defined as the limiting position PT of the secant PQ.



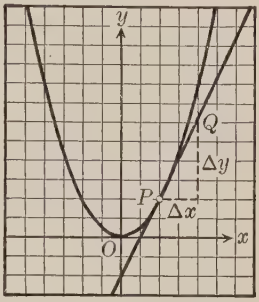
Slope of a Curve. In the figure below, $\frac{\Delta y}{\Delta x}$, which is the average rate of change of y with respect to x in the interval Δx and is also the slope of the secant PQ, may be regarded as the average slope of the curve for the interval Δx. Then as Q moves along the curve, approaching P as its limiting position, PQ also approaches the tangent PT as its limiting position. The average slope $\frac{\Delta y}{\Delta x}$ for the interval Δx therefore approaches the slope of the tangent PT as its limit as Δx approaches zero. The *slope of the curve at the point P* is therefore defined as the slope of the tangent at P.



Since this is also the rate of change of y with respect to x at the instant when a point moving along the curve reaches P, it is called the *instantaneous rate* for the point P.

Instantaneous Rate from Graphs. We can estimate from a graph the approximate instantaneous rate of change of y with respect to x for a given value of x.

For example, to find the instantaneous rate when $x = 2$ in the equation $y = \frac{1}{2}x^2$, we first draw the graph, as here shown, and at the point P, for which $x = 2$, we then draw an approximate tangent.

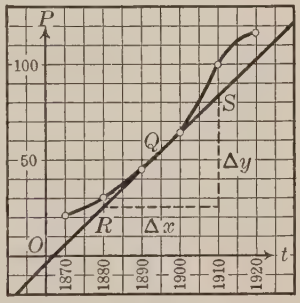


Taking any other point Q on this line and estimating Δy and Δx for these points, we then have

$$\frac{\Delta y}{\Delta x} = \frac{4.1}{2} = 2.05;$$

that is, the slope of the curve $y = \frac{1}{2}x^2$ at $P(2, 2)$, or the instantaneous rate when $x = 2$, is approximately 2.05.

The same method can also be applied in estimating the instantaneous rate of change when the functional relation is expressed by a table of values, such as the following one, which gives the population P, in thousands, of a city for a certain period of years (t):



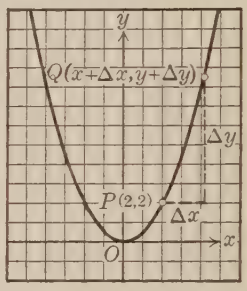
$t =$	1870	1880	1890	1900	1910	1920
$P =$	21	30	45	64	100	118

If we wish to find the instantaneous rate for 1895, we draw the graph and then draw an approximate tangent at Q, for which $t = 1895$. Estimating Δy and Δx for the interval between any two points R and S on the tangent, we have

$$\frac{\Delta y}{\Delta x} = \frac{57}{30} = 1.9;$$

that is, in 1895 the rate of increase was about 1900 a year.

Method of Increments. In the functional relation $y = \frac{1}{2}x^2$, which was considered on page 251, we may employ another method in calculating the approximate instantaneous rate of change of y with respect to x.



When $x = 2$ in the given equation, we have $y = 2$, and hence the point $P(2, 2)$ is one point on the graph. If we then take Q as another point on the graph, we may represent the coordinates of Q by $(2 + \Delta x, 2 + \Delta y)$, where Δx is called the *increment* of the variable x and Δy is the increment of y (or of the function).

Since Q is on the graph, its coordinates satisfy the equation $y = \frac{1}{2}x^2$, and hence we have

$$2 + \Delta y = \tfrac{1}{2}(2 + \Delta x)^2 = 2 + 2\,\Delta x + \tfrac{1}{2}\,\overline{\Delta x}^2,$$

or

$$\Delta y = 2\,\Delta x + \tfrac{1}{2}\,\overline{\Delta x}^2.$$

Dividing both members of this equation by Δx, we have

$$\frac{\Delta y}{\Delta x} = 2 + \tfrac{1}{2}\,\Delta x,$$

which is the average rate of change in the interval Δx.

If we now take a comparatively small interval, for example, if we let $\Delta x = 0.1$, we have

$$\frac{\Delta y}{\Delta x} = 2 + \tfrac{1}{2} \times 0.1 = 2.05;$$

if we take $\Delta x = 0.01$, we have 2.005, and so on for other values of Δx. Any one of these values of $\frac{\Delta y}{\Delta x}$ may be taken as an approximation to the instantaneous rate of change.

These values are also, of course, approximations to the slope of the graph at the point $P(2, 2)$.

Later we shall see that the limiting value of $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$ is 2, and that 2 is the exact instantaneous rate of change when $x = 2$.

Exercises. Increments and Rates

Using the method of increments described on page 252, find the approximate slope of each of the following curves at the specified point on the curve:

1. $y = 0.02\,x^2$; $P(2, 0.08)$.
2. $y = 0.02\,x^2$; $P'(5, 0.5)$.
3. $y = x^3$; $P(2, 8)$.
4. $y = x^3$; $P'(-\frac{1}{2}, -\frac{1}{8})$.
5. $y = \dfrac{1}{x}$; $P(2, 0.5)$.
6. $y = \dfrac{1}{x}$; $P'(-4, -0.25)$.
7. $y = 2\,x^2 + 3$; $P(1, 5)$.
8. $y = 2\,x^2 + 3$; $P'(4, 35)$.
9. $d = 16\,t^2$; $P(2, 64)$.
10. $d = 16\,t^2$; $P(3, 144)$.
11. $V = \dfrac{300}{P}$; $Q(15, 20)$.
12. $V = \dfrac{300}{P}$; $Q'(20, 15)$.

In each of the following equations find the approximate instantaneous rate of change of the dependent variable with respect to the independent variable for the specified value:

13. $y = x^2 + 2\,x$; $x = 1$.
14. $y = x^3 - 1$; $x = 2$.
15. $y = x^3 - x^2 + 2$; $x = 3$.
16. $y^2 = 2\,x^2 + 3$; $x = 0$.
17. $d = 80\,t - 16\,t^2$; $t = 4$.
18. $v = t^3 - 2\,t$; $t = 2$.

19. Draw the graph of the equation $y = 0.05\,x^2$, and from the graph estimate the slope of the curve at the point (10, 5). By the method of increments find approximate values for the slope of the curve at this point, taking the values 0.1, 0.01, and 0.001 for Δx. What is the exact slope of the curve at this point?

20. When a wind is blowing perpendicular to a flat surface, the relation between the pressure P, in pounds per square feet, and the velocity v of the wind, in miles per hour, is expressed by the formula $P = 0.003\,v^2$. Using the method of increments, find the average rate of change of P with respect to v for the interval from $v = 30$ to $v = 40$. Find also an approximate value for the instantaneous rate when $v = 35$.

Idea of Limit. Previous references to limits have involved only ideas that are generally learned in earlier mathematics. The subject, however, is so important in the calculus that it is now desirable to state certain ideas more precisely.

Limit of a Variable. When a variable x approaches a constant c in such a way that the difference between them in absolute value becomes and remains less than any preassigned quantity, however small, x is said *to approach c as a limit.*

This may be represented by the symbols $\lim x = c$ or $x \to c$.

A familiar example of a variable approaching a limit is the area of a regular inscribed polygon. This area approaches the constant area of the circle as the number of sides is increased.

Infinitesimal. If the limit of a variable is zero, the variable is called an *infinitesimal.*

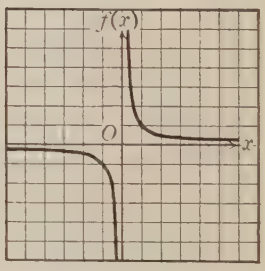
In the definition of the slope of a curve on page 250 both Δx and Δy are infinitesimals, since each approaches zero as a limit.

The reciprocal of an infinitesimal is said to be infinite.

Thus, as $x \to 0$, we see that $\frac{1}{x}$ increases without limit, and hence is said to become infinite as x approaches zero as a limit.

Continuity. In general, the graphs that we have studied thus far have been smooth curves except at special points.

For example, in the graph on page 223 we found when $x \to 2$ that y either increased or decreased without limit, but for other values of x we found that y had a finite value.



In the graph of $f(x) = \frac{1}{x}$ here shown, the function is said to be *discontinuous* when $x = 0$, and *continuous* for all other values of x.

This is one of the most common types of discontinuity. Both a precise definition and a discussion of discontinuity are, however, beyond the scope of our present work.

Properties of Limits. The following properties of limits are important in the work that is to follow:

1. *The limit of the sum of two or more variables is the sum of their respective limits.*

For example, in finding the limit of $3 + \Delta x + \overline{\Delta x}^2$ as $\Delta x \to 0$, we have

$$\lim_{\Delta x \to 0} (3 + \Delta x + \overline{\Delta x}^2) = 3 + \lim_{\Delta x \to 0} \Delta x + \lim_{\Delta x \to 0} \overline{\Delta x}^2 = 3.$$

This and the following properties are to be assumed without proof.

2. *The limit of the product of two or more variables is the product of their respective limits.*

For example, in the case of the limit of $(x + \Delta x)(2x + \Delta x)^2$ we have

$$\lim_{\Delta x \to 0} [(x + \Delta x)(2x + \Delta x)^2] = \lim_{\Delta x \to 0} (x + \Delta x) \cdot \lim_{\Delta x \to 0} (2x + \Delta x)^2 = x \cdot 4x^2 = 4x^3.$$

3. *The limit of the quotient of two variables is the quotient of their respective limits, provided the limit of the divisor is not zero.*

Thus,
$$\lim_{\Delta x \to 0} \frac{2 + \Delta x}{3 + \Delta x + \overline{\Delta x}^2} = \frac{\lim\limits_{\Delta x \to 0} (2 + \Delta x)}{\lim\limits_{\Delta x \to 0} (3 + \Delta x + \overline{\Delta x}^2)} = \frac{2}{3}.$$

Quotient of Infinitesimals. The variables which we shall consider are chiefly the infinitesimals Δx and Δy. Since the limit of the divisor is zero, when we attempt to find the limit of $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$, we cannot apply the third fundamental property given above unless we have a definite relation connecting Δy and Δx. In the calculus, however, we have relations between Δy and Δx which enable us to apply this property. For example, if

$$\Delta y = 2x\Delta x + \overline{\Delta x}^2,$$

we may write
$$\frac{\Delta y}{\Delta x} = 2x + \Delta x.$$

Hence
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x),$$

or
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x.$$

Exercises. Limits

Find the limit in each of the following cases, stating the property or properties of limits involved:

1. $\lim_{\Delta x \to 0} (x + \frac{1}{2}\Delta x)$.
2. $\lim_{\Delta x \to 0} (x^2 - \frac{1}{2}\Delta x)$.
3. $\lim_{\Delta x \to 0} \dfrac{x + \Delta x}{x^2 + \overline{\Delta x}^2}$.
4. $\lim_{\Delta x \to 0} \dfrac{\Delta x}{x + \Delta x}$.
5. $\lim_{\Delta x \to 0} (x + \Delta x)(x - \Delta x)$.
6. $\lim_{\Delta x \to 0} (x^2 + 2\,x\Delta x + \overline{\Delta x}^2)$.
7. $\lim_{\Delta x \to 0} \dfrac{1}{x(x + \Delta x)}$.
8. $\lim_{\Delta x \to 0} \dfrac{(x + \Delta x)(x^2 + \overline{\Delta x}^2)}{2\,x + \Delta x + \overline{\Delta x}^2}$.

In each of Exs. 9–14 find the limit of $\dfrac{\Delta y}{\Delta x}$ *as* $\Delta x \to 0$ *when* Δy *and* Δx *are related as follows:*

9. $\Delta y = x\Delta x + \overline{\Delta x}^2$.
10. $\Delta y = -\dfrac{\Delta x}{x^2 + x\Delta x}$.
11. $\Delta y = \Delta x - \dfrac{\Delta x}{x + \Delta x}$.
12. $\Delta y = 3\,x^2\Delta x + 3\,x\overline{\Delta x}^2$.
13. $\Delta y = 2\,x\Delta x + \dfrac{\Delta x}{x + \Delta x}$.
14. $\Delta y = \Delta x + \dfrac{\Delta x}{x^2 + x\Delta x}$.

15. When $\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$, what is $\lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$?

In finding this limit we may proceed as follows:

$$\Delta y = (\sqrt{x + \Delta x} - \sqrt{x}) \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{x + \Delta x - x}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{\Delta x}{\sqrt{x + \Delta x} + \sqrt{x}}.$$

Find the limit in each of the following cases:

16. $\lim_{x \to 3} \dfrac{3 + 2\,x^2}{1 + x^2}$.
17. $\lim_{x \to 1} \dfrac{1 - x^2}{1 + x^2}$.
18. $\lim_{x \to 2} \dfrac{x^2 - 5\,x + 6}{x - 2}$.
19. $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$.
20. $\lim_{x \to 3} \dfrac{x - 3}{x^2 - 2\,x - 3}$.
21. $\lim_{x \to 0} \dfrac{3\,x^2 - 2\,x}{x}$.
22. $\lim_{x \to 0} \dfrac{2\,x^3 + 5\,x^2}{2\,x^4 - 10\,x^2}$.
23. $\lim_{x \to 3} \dfrac{(x - 2)^2 - 4\,x^3}{x^2 + 24}$.

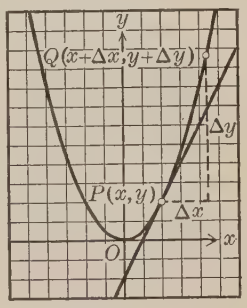
3. The Derivative

The Derivative. The limit of $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$ is called the *derivative of y with respect to x*, and is represented by the symbol $\frac{dy}{dx}$.

It should be observed that $\frac{dy}{dx}$ is a symbol, not a quotient.

To find the derivative in the functional relation $y = \frac{1}{2} x^2$, which was last considered on page 252, we must first find the relation which connects Δy and Δx.

If on the graph of $y = \frac{1}{2} x^2$ we let $P(x, y)$ be any point and Q any other point, the coordinates of Q may be represented by $(x + \Delta x, y + \Delta y)$. Since Q is on the curve, its coordinates satisfy the equation, and hence we have



$$y + \Delta y = \tfrac{1}{2}(x + \Delta x)^2,$$

or $$y + \Delta y = \tfrac{1}{2} x^2 + x\,\Delta x + \tfrac{1}{2}\overline{\Delta x}^2.$$

If we now subtract $y = \frac{1}{2} x^2$, member for member, we have

$$\Delta y = x\,\Delta x + \tfrac{1}{2}\overline{\Delta x}^2,$$

and $$\frac{\Delta y}{\Delta x} = x + \tfrac{1}{2}\Delta x.$$

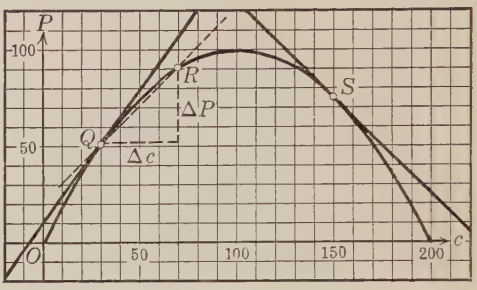
Then, when we let $\Delta x \to 0$ in this last equation, we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = x,$$

or $$\frac{dy}{dx} = x.$$

It should be observed that $\frac{\Delta y}{\Delta x}$ is the average slope of the curve $y = \frac{1}{2} x^2$ for any interval Δx beginning with the point $P(x, y)$, while $\frac{dy}{dx}$ is the exact slope of the curve at P, or the instantaneous rate of change of y with respect to x for any value of x. Thus, if $x = 2$, $\frac{dy}{dx} = 2$; that is, 2 is the limit of $\frac{\Delta y}{\Delta x}$, as stated on page 252.

Differentiation. The process of finding the derivative of a function is called *differentiation*. Before proceeding further it will be helpful to examine more closely the connection between the geometric and algebraic aspects of the work.



For example, considering the derivative of the function $P = 2c - 0.01c^2$ from the geometric point of view, we see that if we draw the graph and take any two points $Q(c, P)$ and $R(c + \Delta c, P + \Delta P)$ on it, $\frac{\Delta P}{\Delta c}$ is the slope of the secant QR, or the average slope of the curve for the interval Δc.

Then $\lim_{\Delta c \to 0} \frac{\Delta P}{\Delta c}$, or $\frac{dP}{dc}$, which is the slope of the tangent at Q, or the slope of the curve at Q, is the instantaneous rate of change of P with respect to c for any value of c.

Since R is a point on the curve, its coordinates satisfy the equation, and hence we proceed algebraically as follows:

$$P + \Delta P = 2(c + \Delta c) - 0.01(c + \Delta c)^2$$

or $$P + \Delta P = 2c + 2\,\Delta c - 0.01c^2 - 0.02\,c\,\Delta c - 0.01\,\overline{\Delta c}^2.$$

Subtracting $P = 2c - 0.01c^2$,

we have $$\Delta P = 2\,\Delta c - 0.02\,c\,\Delta c - 0.01\,\overline{\Delta c}^2,$$

and $$\frac{\Delta P}{\Delta c} = 2 - 0.02\,c - 0.01\,\Delta c.$$

Hence $$\lim_{\Delta c \to 0} \frac{\Delta P}{\Delta c} = \frac{dP}{dc} = 2 - 0.02\,c.$$

Thus, if $c = 30$, $$\frac{dP}{dc} = 2 - 0.02 \times 30 = 1.4;$$

or if $c = 150$, $$\frac{dP}{dc} = 2 - 0.02 \times 150 = -1.$$

That is, the slope at these points is 1.4 and -1 respectively.

Exercises. Derivatives

In each of the following equations find $\frac{dy}{dx}$:

1. $y = x^2$.
2. $y = x^3$.
3. $y = x^4$.
4. $y = \frac{1}{x}$.
5. $y = 1 - x^2$.
6. $y = x^2 - 1$.
7. $y = x^2 - 5x + 6$.
8. $y = \frac{x}{x+1}$.
9. $y = x^3 - 3x^2$.
10. $y = \frac{1}{3}x^3 - x^2 + 2$.
11. $y = x^3 - 1$.
12. $y = \frac{1}{x^2}$.

In each of the following equations find the derivative of the dependent variable with respect to the independent variable:

13. $d = 140t - 16t^2$.
14. $v = 50t - 2t^2$.
15. $F = 120t - 5t^2$.
16. $d = 16t^2$.
17. $y = 0.02x^2$.
18. $A = \pi r^2$.
19. $S = 4\pi r^2$.
20. $V = \frac{1}{6}\pi d^3$.
21. $V = \frac{4}{3}\pi r^3$.

22. In the equation $y = 0.6x - 0.05x^2$ find the instantaneous rate of change of y with respect to x, and find how fast y is changing with respect to x when $x = 1$; when $x = 5$.

23. If the relation between the distance d, in feet, and the time t, in seconds, of a moving body is $d = t^2 - 0.2t$, what is the speed of the body when $t = 2$?

24. The relation between the volume V and the pressure P for a certain gas is given by the formula $V = \frac{300}{P}$. Show that when $P = 10$ the rate of change of V with respect to P is -3, and explain the meaning of the negative sign.

25. In the relation $v = t^2 - 0.4t$ between the speed v, in feet per second, of a car starting from rest and the time t, in seconds, find the acceleration of the car when $t = 2$.

26. Find the rate at which the area of a square is changing per unit of change in the side s when $s = 5$, and also find the rate at which the volume of a cube is changing per unit of change in the edge e when $e = 5$.

Derivatives of Functions. The derivative of a functional relation of the form $y = f(x)$ may be found, as on page 257, by first finding the difference quotient $\frac{\Delta y}{\Delta x}$ and then taking the limit of this quotient when $\Delta x \to 0$. Briefly expressed, the procedure that is followed in applying this method to finding the derivative of $y = f(x)$ is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Instead of applying this method in each case that we need to consider, it is much more convenient to develop general rules for finding the derivatives of various types of functions, or, what is better, to develop formulas which can be applied to the different types, and then to use the formulas in finding the derivatives of the functions.

Just as we can solve quadratic equations of the type $ax^2 + bx + c = 0$, where a, b, and c are constants, by applying the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

so we can find the derivatives of certain types of functions by applying the formulas that we shall now develop.

General Types of Functions. A function which can be formed by applying to its variable a finite number of times one or more of the operations of addition, subtraction, multiplication, division, and the finding of powers and roots is called an *algebraic function*, while one that is not algebraic is called a *transcendent function*.

For example, $\frac{1}{2}x^3$, $x^2 + 1$, $\sqrt{\frac{1}{x}}$, $\frac{\sqrt{x} + 5}{x^3 - 3}$, $x^3 - \frac{1}{x^2}$, and $x^{\frac{2}{3}} - x^{-\frac{2}{3}}$ are all algebraic functions of x, while $\sin x$, $\csc x$, $\log x$, e^x, and $\tan^{-1} x$ are transcendent functions of x.

Although in our work in trigonometry we considered certain transcendent functions, and in our work in analytic geometry we considered several types of exponential functions, in our subsequent work in the calculus we shall consider only simple types of algebraic functions.

Derivative of the Power Function. The function $y = x^n$, where n is either a positive or a negative integer, is called a *power function* of x.

We may deduce a formula for finding the derivative of this function when n is a positive integer by substituting $x + \Delta x$ for x and $y + \Delta y$ for y, which gives us

$$y + \Delta y = (x + \Delta x)^n.$$

Expanding by the Binomial Theorem, we then have

$$y + \Delta y = x^n + nx^{n-1}\Delta x + \tfrac{1}{2}\, n(n-1)x^{n-2}\overline{\Delta x}^2 + \tfrac{1}{6}\, n(n-1)(n-2)x^{n-3}\overline{\Delta x}^3 + \cdots.$$

Then, subtracting $y = x^n$, we have

$$\Delta y = nx^{n-1}\Delta x + \tfrac{1}{2}\, n(n-1)x^{n-2}\overline{\Delta x}^2 + \tfrac{1}{6}\, n(n-1)(n-2)x^{n-3}\overline{\Delta x}^3 + \cdots;$$

from which, dividing by Δx, we have

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \tfrac{1}{2}\, n(n-1)x^{n-2}\Delta x + \tfrac{1}{6}\, n(n-1)(n-2)x^{n-3}\overline{\Delta x}^2 + \cdots.$$

Then, by the first property of limits stated on page 255,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} nx^{n-1} + \lim_{\Delta x \to 0} \tfrac{1}{2}\, n(n-1)x^{n-2}\Delta x + \lim_{\Delta x \to 0} \tfrac{1}{6}\, n(n-1)(n-2)x^{n-3}\overline{\Delta x}^2 + \cdots.$$

Since every term after the first contains Δx as a factor, we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = nx^{n-1},$$

or

$$\frac{dy}{dx} = nx^{n-1},$$

where n is a positive integer, or, expressed in words,

The derivative of any positive integral power of a variable is the product of the exponent and the next lower integral power of the variable.

For example, when $y = x^{100}$, we have $\frac{dy}{dx} = 100\, x^{99}$. It would obviously be much longer to evaluate $(x + \Delta x)^{100}$, as in the general method.

Formula for $y = kx^n$. The formula for the function $y = kx^n$, where k is a constant, is also easily deduced. Substituting $x + \Delta x$ for x and $y + \Delta y$ for y, we have

$$y + \Delta y = k(x + \Delta x)^n,$$

from which, proceeding as on page 261, we have

$$\Delta y = knx^{n-1}\Delta x + \tfrac{1}{2}\, kn(n-1)x^{n-2}\overline{\Delta x}^2 + \cdots,$$

and

$$\frac{\Delta y}{\Delta x} = knx^{n-1} + \tfrac{1}{2}\, kn(n-1)x^{n-2}\Delta x + \cdots.$$

Hence, in finding the derivative, we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [knx^{n-1} + \tfrac{1}{2}\, kn(n-1)x^{n-2}\Delta x + \cdots],$$

or

$$\frac{dy}{dx} = knx^{n-1}.$$

For example, when $y = 3x^4$, we have $\frac{dy}{dx} = 12x^3$, and when $A = \pi r^2$, we have $\frac{dA}{dr} = 2\pi r$.

Negative and Fractional Powers. We have shown that the formula

$$\frac{dy}{dx} = nx^{n-1}$$

gives the derivative of $y = x^n$, where n is a positive integer, and that the formula

$$\frac{dy}{dx} = knx^{n-1}$$

gives the derivative of $y = kx^n$, where k is any constant and n is a positive integer. These formulas are equally valid for negative and fractional values of n.

For example, when $y = 2x^{\frac{1}{2}}$, we have $\frac{dy}{dx} = 2 \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}} = x^{-\frac{1}{2}}$, or $\frac{1}{\sqrt{x}}$; and when $V = \frac{300}{P}$, we have $V = 300P^{-1}$, and so $\frac{dV}{dP} = -300P^{-2} = -\frac{300}{P^2}$.

The proof of these formulas for negative and fractional powers is, however, beyond the scope of our present work, and hence for our purposes we shall simply assume them to be valid.

Exercises. Power Functions

By means of the formulas find $\frac{dy}{dx}$ *when*

1. $y = x^6$.
2. $y = x^{-4}$.
3. $y = 6\,x^{\frac{1}{2}}$.
4. $y = \frac{10}{x^2}$.
5. $y = 2\,x^{\frac{2}{3}}$.
6. $y = \frac{3}{4}\,x^{\frac{3}{4}}$.
7. $y = 4\sqrt[5]{x^3}$.
8. $y = \frac{10}{\sqrt{x}}$.
9. $y = \frac{1}{2}\sqrt{x^3}$.
10. $y = 5\,x^{\frac{3}{5}}$.
11. $y = x^{-1}$.
12. $y = \frac{3}{\sqrt[3]{x}}$.

Find the derivative of each of the following functions:

13. $3\,x^2$.
14. $5\,x^3$.
15. $\frac{1}{x^2}$.
16. $2\sqrt{x}$.
17. $\frac{1}{4}\,\pi d^2$.
18. $\frac{1}{x^3}$.
19. $2\,\pi r$.
20. $4\,\pi r^2$.
21. $\frac{2}{x^3}$.
22. $\frac{4}{3}\,\pi r^3$.
23. $\frac{1}{2}\,gt^2$.
24. $\frac{10}{P}$.

In Ex. 23 take g as a constant and t as the variable.

25. From the formula $V = \frac{1}{6}\,\pi d^3$, which gives the volume V of a sphere in terms of the diameter d, find the rate at which V is changing when $d = 7$; when $d = 35$.

26. When the intensity I of light on an object is related to the distance x to the object as shown by the formula $I = \frac{1500}{x^2}$, what is the rate of change in the intensity with respect to distance when $x = 20$?

27. The electric resistance R of a certain kind of wire in terms of the diameter x of the wire is given by the formula $R = \frac{88.5}{x}$. Find $\frac{dR}{dx}$, and then find the rate of change of R with respect to x when $x = 3$.

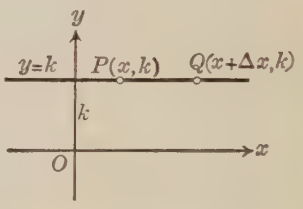
28. The relation between the acceleration g imparted to a body by gravity and the distance x, in miles, of the body from the earth's center may be expressed as $g = 5.12 \times 10^8 x^{-2}$. From this formula find the derivative which shows how rapidly g is changing when $x = 4010$.

Derivative of a Constant. The graph of the equation $y = k$, where k is a constant, is a straight line parallel to the x axis. Hence, if we take $P(x, k)$ as one point on the graph, the ordinate for any other point, as Q for which the abscissa is $x + \Delta x$, does not change; that is, $\Delta y = 0$, and hence

$$\frac{\Delta y}{\Delta x} = 0.$$

Since the graph is a straight line, its slope is constant, and hence the slope of the line at P is zero; that is, we may write

$$\frac{dy}{dx} = 0.$$



Then, since $y = k$, we may write

$$\frac{dk}{dx} = 0.$$

This formula shows that

The derivative of a constant with respect to a variable is 0.

Derivative of a Variable with Respect to Itself. In the same way, if we let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be any two points on the graph of the equation $y = x$, which is the straight line here shown, we have

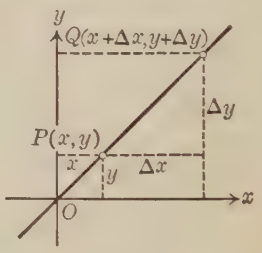
$$y + \Delta y = x + \Delta x.$$

Subtracting $y = x$,

we have $\Delta y = \Delta x$;

whence $\frac{\Delta y}{\Delta x} = 1.$

Then, as above, $\frac{dy}{dx} = 1,$



or, since $y = x$, we may write this last equation in the form

$$\frac{dx}{dx} = 1.$$

That is, expressed in words, this formula shows that

The derivative of a variable with respect to itself is 1.

Derivative of the Sum of Two Functions. A general expression for the sum of two functions is $y = u + v$, where both u and v are functions of the variable x.

Thus, if $u = x^3$ and $v = 3\,x^4$, we have $y = x^3 + 3\,x^4$.

Then, if x takes an increment Δx, u and v will take the increments Δu and Δv respectively, and hence, applying the general method of page 257, we have

$$y + \Delta y = u + \Delta u + v + \Delta v.$$

Then, subtracting $y = u + v,$

we have $$\Delta y = \Delta u + \Delta v,$$

from which $$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

Now, by the first property of limits on page 255, we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x},$$

or $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

Then, since $$y = u + v,$$

we have $$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

Expressed in words, this formula shows that

The derivative of the algebraic sum of two functions is the algebraic sum of their derivatives.

For example, if $y = 3\,x^3 + 3\,x^4$, we have $\frac{dy}{dx} = 9\,x^2 + 12\,x^3$.

Sum of Several Functions. The above formula is readily extended in the same way to apply to any number of functions; that is, if $y = u + v - w$, we have

$$\frac{d(u+v-w)}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}.$$

Thus, if $y = x^2 + 5\,x^3 - 4\,x^5$, we have $\frac{dy}{dx} = 2\,x + 15\,x^2 - 20\,x^4$, and if $y = x^4 + 3\,x^2 - 2\,x + 5$, we have $\frac{dy}{dx} = 4\,x^3 + 6\,x - 2$.

Derivative of the Product of Two Functions. As a general expression for the product of two functions we may take $y = uv$, where both u and v are functions of x. If we then proceed in the same way as on page 265, we have

$$y + \Delta y = (u + \Delta u)(v + \Delta v),$$

or $$y + \Delta y = uv + u\Delta v + v\Delta u + \Delta u\Delta v.$$

Then, subtracting $y = uv$,

we have $$\Delta y = u\Delta v + v\Delta u + \Delta u\Delta v;$$

whence $$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}.$$

Now, by the first property of limits on page 255, we have

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} u\frac{\Delta v}{\Delta x} + \lim_{\Delta x \to 0} v\frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0}\Delta u\frac{\Delta v}{\Delta x},$$

and, by the second property, we have

$$\lim_{\Delta x \to 0}\Delta u\frac{\Delta v}{\Delta x} = \lim_{\Delta x \to 0}\Delta u \cdot \lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x} = 0 \cdot \frac{dv}{dx} = 0.$$

Hence $$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Expressed in words, the above formula shows that

The derivative of the product of two functions is the product of the first function and the derivative of the second plus the product of the second and the derivative of the first.

Product of a Constant and a Function. If in the above discussion we replace u by a constant k, the formula becomes

$$\frac{d(kv)}{dx} = k\frac{dv}{dx},$$

the derivative of u with respect to x being zero since it becomes the derivative of the constant k. That is,

The derivative of the product of a constant and a function is the product of the constant and the derivative of the function.

Exercises. Derivatives of Sums and Products

Using the appropriate formula, find the derivative of each of the following functions:

1. $y = x^3 - 2x^2 + 5$.
2. $y = 3x^4 - 2x^3 + 2$.
3. $d = 120t - 16t^2$.
4. $d = 25 + 120t - 16t^2$.
5. $v = 3t^3 - 2t^2 - 0.6t$.
6. $y = x^4 + 3x^3 - 2x^2 + 1$.
7. $h = 6000 - 16t^2$.
8. $v = 32t + 50$.
9. $v = 4t^3 - 0.6t^2$.
10. $R = 5x^2 - 6x + 7$.
11. $P = 2c - 0.01c^2$.
12. $E = 4x^3 - 0.5x^2 + 0.06x$.

In each of the following cases find $\frac{dy}{dx}$*:*

13. $y = (x^2 + 1)(x^3 - 5)$.
14. $y = 16(x^3 - 16x^2 + 5)$.
15. $y = (x+2)(x^2+3x+6)$.
16. $y = 4(x^3 - 6x^2)$.
17. $y = (x^2 + 2x)(x^2 - 2x)$.
18. $y = x^3(x^2 + 6x + 4)$.

19. Taking $y = uvw$ as a general expression for the product of three functions of x, prove by the method used on page 266 that $\frac{d(uvw)}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$.

20. From the formula given in Ex. 19 find the derivative of $y = x^2(x^3 + 1)(x - 1)$. Then perform the indicated multiplication and find the derivative of the product. Compare the two results.

21. Find the derivative of $y = \frac{1}{3}x^3 - x - 2$, and compute its value when x has the values 0, ± 1, ± 2, and ± 3. Draw the graph of the curve, and on the same axes draw the graph of the derivative.

22. Find the slope of the curve $y = x^3 - 3x + 2$ at the point for which $x = 0$; $x = \frac{1}{2}$; $x = 1$; $x = 2$.

23. Find the slope of the curve $y = \frac{1}{3}x^3 - 3x + 2$ at the point for which $x = \frac{1}{2}$; $x = 1$; $x = 1\frac{1}{2}$; $x = 2$.

Power of a Function. When considering the power function $y = x^n$ on page 261, we found that

$$\frac{dy}{dx} = nx^{n-1}.$$

This law is also true for a power of any algebraic function of x. Although the complete proof is beyond the scope of our present work, we can readily show that the law is valid in the case of $y = u^n$, where u is any algebraic function of x.

Letting $u = v$ in the product formula on page 266, we have

$$\frac{d(u \cdot u)}{dx} = u\frac{du}{dx} + u\frac{du}{dx},$$

or

$$\frac{d(u^2)}{dx} = 2u\frac{du}{dx}.$$

Similarly, letting $u = v = w$ in the formula given in Ex. 19 on page 267, we have

$$\frac{d(u \cdot u \cdot u)}{dx} = u \cdot u\frac{du}{dx} + u \cdot u\frac{du}{dx} + u \cdot u\frac{du}{dx},$$

or

$$\frac{d(u^3)}{dx} = 3u^2\frac{du}{dx}.$$

We shall now assume without proof that

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$$

That is, expressed in words, this formula shows that

The derivative of a function raised to any power is the product of (1) *the exponent,* (2) *the function raised to a power 1 less than the given power, and* (3) *the derivative of the function with respect to its variable.*

For example, when $y = (x^2 + 2x + 5)^4$, we have

$$\begin{aligned}\frac{dy}{dx} &= 4(x^2 + 2x + 5)^3 \cdot \frac{d(x^2 + 2x + 5)}{dx} \\ &= 4(x^2 + 2x + 5)^3(2x + 2) \\ &= 8(x + 1)(x^2 + 2x + 5)^3.\end{aligned}$$

Derivative of the Quotient of Two Functions. Taking $y = \frac{u}{v}$, where u and v are functions of x, as a general expression for the quotient of two functions, we have, as before,

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}.$$

Then, subtracting $y = \frac{u}{v}$,

we have $$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\,\Delta u - u\,\Delta v}{v^2 + v\,\Delta v}.$$

Dividing by Δx in the usual way, we then have

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v^2 + v\,\Delta v},$$

from which, by the second and third properties of limits given on page 255, we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v^2 + v\,\Delta v},$$

or $$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Expressed in words, this formula shows that

The derivative of the quotient of two functions is the product of the denominator and the derivative of the numerator minus the product of the numerator and the derivative of the denominator, all divided by the square of the denominator.

For example, when $y = \frac{x^3 + 2x}{x^2 + 1}$, we have $u = x^3 + 2x$ and $v = x^2 + 1$

Now $$\frac{du}{dx} = 3x^2 + 2 \quad \text{and} \quad \frac{dv}{dx} = 2x.$$

Hence, by substituting in the above formula, we have

$$\frac{dy}{dx} = \frac{(x^2 + 1)(3x^2 + 2) - (x^3 + 2x) \cdot 2x}{(x^2 + 1)^2}$$

$$= \frac{x^4 + x^2 + 2}{(x^2 + 1)^2}.$$

Derivatives of Radical Expressions. In applying the formulas that were developed on pages 265–269 to functions in which the radical sign appears, it is first necessary to write the given expression with fractional exponents, as shown in the following examples:

1. Find the derivative of $y = \sqrt{9 - x^2}$.

That is, in finding the derivative of $y = (9 - x^2)^{\frac{1}{2}}$, in which $u = 9 - x^2$ and $n = \frac{1}{2}$, we first find that

$$\frac{du}{dx} = -2x,$$

and hence $$\frac{dy}{dx} = \tfrac{1}{2}(9 - x^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{9 - x^2}}.$$

After a little practice it will not be necessary to write down the values of u and n, and the substitutions may be made directly in the formula.

2. Find the derivative of $y = \dfrac{x^2}{\sqrt{2x + 1}}$.

Using fractional exponents and substituting directly in the formula on page 269, we have

$$\frac{dy}{dx} = \frac{(2x + 1)^{\frac{1}{2}} \cdot \frac{d(x^2)}{dx} - x^2 \cdot \frac{d(2x + 1)^{\frac{1}{2}}}{dx}}{2x + 1}$$

$$= \frac{(2x + 1)^{\frac{1}{2}} \cdot 2x - x^2 \cdot \frac{1}{2}(2x + 1)^{-\frac{1}{2}} \cdot 2}{2x + 1}$$

$$= \frac{2x(2x + 1) - x^2}{(2x + 1)^{\frac{3}{2}}}$$

$$= \frac{3x^2 + 2x}{\sqrt{(2x + 1)^3}}.$$

Exercises. Derivatives Involving Powers and Quotients

Find the derivative of each of the following functions:

1. $(x^2 + 2)^3$.
2. $(2x + 1)^5$.
3. $\sqrt{x^2 + 3x}$.
4. $\dfrac{\sqrt{x + 5}}{2x + 3}$.
5. $(x^2 + 5x)^2$.
6. $\sqrt{x + 1}$.
7. $\sqrt{x^3 - 1}$.
8. $\dfrac{x^2 + 1}{x^2 - 1}$.
9. $(x^3 + 3x^2 + 5)^{\frac{3}{2}}$.
10. $\sqrt{(3x^2 + 5)^3}$.
11. $\sqrt{x^2 + 1} - \sqrt{1 - x^2}$.
12. $\sqrt{\dfrac{x + 1}{x - 1}}$.

Summary of Formulas. For greater convenience the different formulas for derivatives that we have developed are here set forth, together with references to the pages on which the statements of the rules appear.

1. The derivative of a constant (k) with respect to a variable:

$$\frac{dk}{dx} = 0.$$ Page 264

2. The derivative of a variable with respect to itself:

$$\frac{dx}{dx} = 1.$$ Page 264

3. The derivative of the algebraic sum of two functions:

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$ Page 265

4. The derivative of the product of two functions:

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$ Page 266

5. The derivative of the product of a constant (k) and a function:

$$\frac{d(kv)}{dx} = k\frac{dv}{dx}.$$ Page 266

6. The derivative of a power of a variable:

$$\frac{d(x^n)}{dx} = nx^{n-1},$$ Page 261

and

$$\frac{d(kx^n)}{dx} = knx^{n-1}.$$ Page 262

7. The derivative of a function raised to any power:

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$$ Page 268

8. The derivative of the quotient of two functions:

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$ Page 269

Exercises. Differentiation

Find the derivative of each of the following functions:

1. $y = x\sqrt{2}$.
2. $y = 3 + x^2$.
3. $y = (3 + x)^2\sqrt{3}$.
4. $y = (x - 2)^2$.
5. $y = x^2 + 4x + 4$.
6. $y = 9 - x(3 - x)$.
7. $y = (x + 1)(x - 1)$.
8. $y = x^2 - 16$.
9. $y = x^2 - 4x - \sqrt{x}$.
10. $y = 5x^4 + 6x^3 - 3x^2$.
11. $y = ax^2 + bx + c$.
12. $d = a + bt + ct^2$.
13. $y = x^{\frac{3}{2}} + 5x^{\frac{1}{2}} + 2$.
14. $y = 6x^{\frac{5}{2}} - 3x^{-\frac{1}{2}}$.
15. $y = \sqrt[3]{x^2} + \sqrt{x} + 5$.
16. $y = (x + 1)(x + 2)^2$.
17. $y = \dfrac{x}{3 + x}$.
18. $y = \dfrac{2}{\sqrt{x} - 1}$.
19. $V = \dfrac{240}{P}$.
20. $y = 2\sqrt{x} + \dfrac{3}{\sqrt{3}}$.
21. $d = \dfrac{t^2 + 2t + 5}{t^2 - 4}$.
22. $d = \sqrt{t^2 + 2}$.
23. $y = (x^2 + 2x + 6)^5$.
24. $y = \sqrt{r^2 - x^2}$.
25. $y = \sqrt{x^2 - r^2}$.
26. $W = (3 - x)^{\frac{1}{3}}$.
27. $y = (3 - x)^{-1}$.
28. $y = (x - 1)^{-\frac{1}{2}}$.
29. $y = (x - 3)(3 - x)$.
30. $y = 3k(x^2 + 1) + x$.
31. $y = \frac{2}{3}(x^2 - 1) - x$.
32. $y = 3kx^3 + 4x^2 - 2x + 5$.
33. $y = \frac{2}{3}x^3 + \frac{1}{2}x^2 + x + 2$.
34. $d = a + v_0t - \frac{1}{2}gt^2$.
35. $y = x^{\frac{1}{4}} + x^{\frac{1}{3}} - 2x^{\frac{1}{2}} + 1$.
36. $y = 3x^{-1} + 2x^{-2} - 3$.
37. $y = \sqrt{x} - \sqrt[3]{x^2} + 1$.
38. $y = \dfrac{x^4 - 1}{x + 1}$.
39. $y = \dfrac{1}{2(x^2 + 3)}$.
40. $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$.
41. $y = \dfrac{x}{x^2 + 2x - 1}$.
42. $y = \dfrac{2x}{x^2 + 2x - 6}$.

The letters a, b, c, g, k, r, and v_0, above, represent constants.

43. The distance d, in feet, over which a certain particle moves in t seconds is given by the formula $d = 6t^2 - 0.2t$. Find the speed (the rate of change of the distance d per unit of time t) of the particle when $t = 3$.

44. Under certain conditions the formula $d = 75t - 8t^2$ gives the distance d, in feet, reached in t seconds by a ball rolling up an inclined plane. Find an expression for the speed of the ball for any value of t, and then find this speed when $t = 3$; when $t = 1\frac{1}{2}$; when $t = 5$.

45. The formula $h = 1500t - 16t^2$ shows the height h reached in t seconds by a projectile fired vertically upward at a certain velocity. Find (1) an expression for the speed of the projectile for any value of t, (2) the initial velocity of the projectile (that is, for $t = 0$), and (3) the velocity for $t = 3$.

46. The resistance R, in pounds per ton of weight, of a train that is traveling v miles per hour is given approximately by the formula $R = 6 + 0.009v^2$. Find the rate of change of R with respect to v when $v = 25$.

47. The viscosity V of water at a temperature of t degrees centigrade is given by the formula $V = \dfrac{0.0157}{1 + 0.035t}$. How fast is the viscosity changing with respect to the temperature when $t = 12$?

48. The formula $E = 21{,}483 - 0.295t - 0.02415t^2$ expresses the relation between the modulus of elasticity E of steel in kilograms per square millimeter and the temperature t in degrees centigrade. Find $\dfrac{dE}{dt}$ when $t = 150$.

49. The pressure P of 1 lb. of saturated steam in terms of its volume V, in cubic feet, is given by the equation $P = \dfrac{478.2}{V^{1.064}}$. Find $\dfrac{dP}{dV}$ when $V = 26.43$; when $V = 200$; when $V = 500$.

Successive Derivatives. By finding the derivative of the function

$$y = x^4 - 2x^3 + x^2,$$

we find the slope of the curve which the function represents; that is, this slope is given by the equation

$$\frac{dy}{dx} = 4x^3 - 6x^2 + 2x. \tag{1}$$

If we let m represent this slope, then m is a function of x, or

$$m = 4x^3 - 6x^2 + 2x.$$

By finding the derivative of this function we can find the rate at which the slope is changing with respect to x; that is, we have

$$\frac{dm}{dx} = 12x^2 - 12x + 2. \tag{2}$$

Representing this rate by r, we then have

$$r = 12x^2 - 12x + 2,$$

and by finding the derivative of this function we have

$$\frac{dr}{dx} = 24x - 12. \tag{3}$$

In (1) we have the derivative of the function with respect to x; in (2) we have the derivative of this derivative, or the *second derivative*; and in (3) we have the derivative of the second derivative, or the *third derivative*. We may also have further derivatives. Either of the following schemes of notation is convenient in designating successive derivatives:

$$y = x^4 - 2x^3 + x^2, \quad \text{or} \quad f(x) = x^4 - 2x^3 + x^2;$$

$$\frac{dy}{dx} = 4x^3 - 6x^2 + 2x, \quad \text{or} \quad f'(x) = 4x^3 - 6x^2 + 2x;$$

$$\frac{d^2y}{dx^2} = 12x^2 - 12x + 2, \quad \text{or} \quad f''(x) = 12x^2 - 12x + 2;$$

$$\frac{d^3y}{dx^3} = 24x - 12, \quad \text{or} \quad f'''(x) = 24x - 12;$$

$$\frac{d^4y}{dx^4} = 24, \quad \text{or} \quad f^{\text{iv}}(x) = 24.$$

Exercises. Successive Differentiation

In each of the following find the first, second, and third derivatives of y with respect to x:

1. $y = x^3 - 2x^2 + 4x + 5$.
2. $y = x^4 - 2x^3 + 3$.
3. $y = x^2 + 2x + 5$.
4. $y = x^4 - 0.2x^2$.
5. $y = x^{\frac{5}{2}} - x^{\frac{3}{2}}$.
6. $y = (x+1)^3$.

7. In $f(x) = x^3 - 6x^2 - 5x + 2$ find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{iv}(x)$ and find for what values of x we have $f'(x) = 0$.

8. If the distance d, in feet, through which a body moves in t seconds is given by the equation $d = 3t^2 - 2t$, what is the speed of the body when $t = 3$? What is the acceleration (that is, at what rate is the speed changing with respect to t)?

We have $f'(t) = 6t - 2$ and $f''(t) = 6$. That is, when $t = 3$, the speed is 16 ft./sec., and the acceleration, which is constant, is 6 ft./sec.^{2} (read "6 ft. per second per second").

As in Ex. 8, find the speed and the acceleration of a moving body for the specified values of t in each of the following cases:

9. $d = 400t - 16t^2$; $t = 3$.
10. $d = 50 + 360t - 16t^2$; $t = 0$, $t = 2$, and $t = 4$.
11. $d = t^3 + 2t^2 + t$; $t = 2$.
12. $d = t^3 - 2t^2 + t$; $t = 2$ and $t = 3$.

13. In each of Exs. 9–12, above, find the rate of change of the acceleration of the body.

14. As shown in Ex. 19 on page 160, the horizontal distances x in the flight of a projectile are given by the equation $x = v_0 t \cos\alpha$. Find $\frac{dx}{dt}$ when $\alpha = 36°$ and $v_0 = 2000$ ft./sec.

15. In Ex. 14 the heights y are given by the equation $y = v_0 t \sin\alpha - \frac{1}{2}gt^2$. Taking $v_0 = 2000$ ft./sec., $\alpha = 36°$, and $g = 32.16$, find $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$, and interpret the results.

16. In the formula $V = \frac{300}{P}$ find $\frac{dV}{dP}$ and $\frac{d^2V}{dP^2}$ when $P = 5$.

17. The distance d, in feet, traveled in t seconds by a certain particle is given by the formula $d = 16\,t^3 - t^4$. Find the speed and the acceleration when $t = 1\frac{1}{2}$.

18. The deflection y measured at a distance x from the fixed end of a certain uniform beam is given by the formula $y = 0.003\,(5\,x^2 - \frac{1}{6}\,x^3)$. Find $\frac{dy}{dx}$ when $x = 5$.

19. The distance d, in feet, through which a certain particle moves in t seconds is given by the formula $d = 16\,t^2 - 20\,t + 10$. Find an expression for the velocity and the acceleration at any time t. What is the speed of the particle when $t = 2$?

20. A body is moving in a straight line in such a way that its distance d, in feet, from a fixed point on the line is given by the formula $d = 1.1 + 2.2\,t + 4.3\,t^2$, where t is the time in seconds. The momentum M of the body is given by the formula $M = \frac{wv}{g}$, where w is the weight and v is the speed, and the kinetic energy K is given by the formula $K = \frac{wv^2}{2\,g}$. Taking $g = 32$, find the momentum and the kinetic energy at the end of 2 sec. of a body weighing 100 lb.

21. Experiments have shown that, for a certain range of temperature, the total amount of heat Q required to raise the temperature of 1 g. of water from 0° to θ degrees centigrade, is given by the formula $Q = \theta + 2 \cdot 10^{-5}\theta^2 + 3 \cdot 10^{-7}\theta^3$. Find the derivative of Q with respect to θ.

22. Taking specific heat at a given temperature as the rate of increase in the quantity of heat per unit of rise in temperature, find the specific heat in Ex. 21 when $\theta = 50°$.

23. For the function $y = ax^2 + bx + c$ show that the second derivative of y with respect to x is negative when a is negative and positive when a is positive.

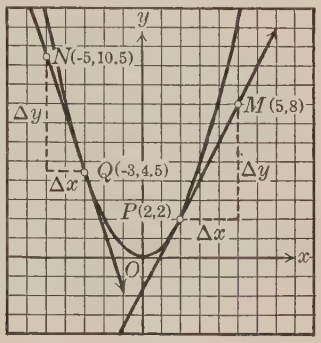
4. Tangents and Normals

Drawing a Tangent. Now that we have learned how to calculate the slope of a curve at any point, which is the slope of the tangent at that point, it is easy to draw a tangent. For example, in the equation

$$y = \tfrac{1}{2}x^2,$$

we have $\dfrac{dy}{dx} = x,$

and hence at the point P, for which $x = 2$, the slope m of the tangent is 2.



To draw the tangent at P, we may then take a point M such that $\dfrac{\Delta y}{\Delta x} = \dfrac{6}{3} = 2$, as shown in the figure, and draw the tangent through P and M.

Similarly, for the tangent at $Q(-3, 4.5)$, we have $m = \dfrac{\Delta y}{\Delta x} = -3$. Taking a point N such that $\Delta y = 6$ and $\Delta x = -2$ (or using other convenient lengths), we may then draw the tangent through Q and N.

From the graph it is easily seen that

When the derivative of y with respect to x is positive, y increases as x increases, and when the derivative is negative, y decreases as x increases.

Exercises. Tangents

In each of the following cases find the slope of the curve at the point specified, determine whether y is increasing or decreasing, and draw the tangent to the curve at P:

1. $y = x^3 - 3x^2 + x$; $P(1, -1)$.
2. $y = x^2 - 2x + 5$; $P(3, 8)$.
3. $y = -x^2 + 2x + 4$; $P(-1, 1)$.
4. $y = x^3 - x^2$; $P(2, 4)$.
5. $y = x^3 + x$; $P(1, 2)$.
6. $y = -x^2 + 4$; $P(2, 0)$.

Normal to a Curve. A line that is perpendicular to a tangent to a curve at the point of tangency is called a *normal* to the curve.

In the figure below, the line PN is the normal at P because it is perpendicular to the tangent at P.

Finding the Equations of Tangents and Normals. The following examples illustrate how the derivative may be used in finding the equations of the tangent and the normal to a curve at a specified point:

1. Find the equation of the tangent at the point $P(2, 2)$ to the parabola $y = \frac{1}{2}x^2$.

Proceeding in the same way as on page 277, we first find that

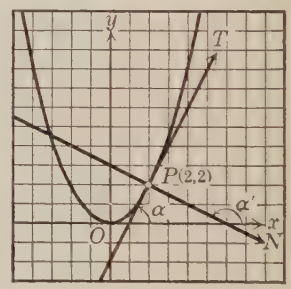
$$\frac{dy}{dx} = x.$$

For the given point $P(2, 2)$ we have $x = 2$, and hence the slope m of the tangent PT is 2.

Using the point-slope formula $y - y_1 = m(x - x_1)$, which was given on page 186, we then have

$$y - 2 = 2(x - 2),$$

or $$2x - y - 2 = 0.$$



Since this is the equation of the line through the point $P(2, 2)$ having the slope 2, it is the required equation of the tangent to the curve at P.

By assigning a convenient value to x, we may now locate a second point, as (5, 8), through which we can draw the tangent PT.

2. In Ex. 1 find the equation of the normal to the curve at the point P.

Since the normal is perpendicular to the tangent, the slope m' of the normal is the negative reciprocal of the slope m of the tangent; that is,

$$m' = -\frac{1}{m} = -\frac{1}{2},$$

and hence, from the point-slope formula, we find that the equation of the normal PN is

$$y - 2 = -\tfrac{1}{2}(x - 2),$$

or $$x + 2y - 6 = 0.$$

3. Find the equations of the tangent and the normal at the point $P(4, 4)$ to the parabola $y^2 = 4x$.

Solving for y, so that we may differentiate for y with respect to x, we have

$$y = 2\sqrt{x} = 2x^{\frac{1}{2}},$$

from which

$$\frac{dy}{dx} = 2 \cdot \tfrac{1}{2} x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}.$$

Hence at the point P, for which $x = 4$, the slope m of the tangent is $\frac{1}{2}$. Substituting in the point-slope formula, we then have

$$y - 4 = \tfrac{1}{2}(x - 4),$$

or

$$x - 2y + 4 = 0,$$

which is the required equation of the tangent to the parabola at the point $P(4, 4)$.

For the slope m' of the normal at P we then have

$$m' = -\frac{1}{m} = -2,$$

and hence, from the point-slope formula, we have

$$y - 4 = -2(x - 4),$$

or

$$2x + y - 12 = 0,$$

which is the required equation of the normal to the curve at $P(4, 4)$.

The student should draw a figure for this example similar to the one for Exs. 1 and 2 on page 278.

4. Find the equation of the tangent at the point $P(1, 1.5)$ to the ellipse $3x^2 + 4y^2 = 12$.

First solving for y, as in Ex. 3 above, we have

$$y = \tfrac{1}{2}\sqrt{12 - 3x^2} = \tfrac{1}{2}(12 - 3x^2)^{\frac{1}{2}},$$

from which

$$\frac{dy}{dx} = \tfrac{1}{2} \cdot \tfrac{1}{2}(12 - 3x^2)^{-\frac{1}{2}} \cdot (-6x) = \frac{-3x}{2\sqrt{12 - 3x^2}}.$$

Hence for the slope m of the tangent at $P(1, 1.5)$ we have

$$m = \frac{-3}{2\sqrt{12 - 3}} = -\frac{3}{2 \cdot 3} = -\frac{1}{2}.$$

Then, substituting in the point-slope formula, we have

$$y - 1.5 = -\tfrac{1}{2}(x - 1),$$

or

$$x + 2y - 4 = 0,$$

which is the required equation of the tangent to the given ellipse at the point $P(1, 1.5)$.

As in Ex. 3, the student should draw a figure to illustrate this case.

Exercises. Tangents and Curves

In Exs. 1–12 find the equation of the tangent to the curve at the specified point, plot the curve, and draw the tangent:

1. $y^2 = 8x$; $P(2, 4)$.
2. $y^2 = -3x$; $P(-3, 3)$.
3. $y^2 = -3x$; $P(-3, -3)$.
4. $y = 16x^2$; $P(-\frac{1}{4}, 1)$.
5. $y = -4x^2$; $P(-1, -4)$.
6. $x^2 + y^2 = 25$; $P(-3, 4)$.
7. $y^2 = 4x$; $P(4, 4)$.
8. $y^2 = 4x$; $P(\frac{1}{4}, 1)$.
9. $y^2 = 4x$; $P(0, 0)$.
10. $x^2 - y^2 = 16$; $P(5, 3)$.
11. $16x^2 + 9y^2 = 25$; $P(1, 1)$.
12. $25x^2 - 9y^2 = 16$; $P(1, 1)$.

13. Find the slope of the curve $y = x^3 - 3x^2 + 2x$ at the point for which $x = 2$, and write the equation of the tangent at this point.

14. Draw the graph of the equation $y = x^3 - 3x^2$, and find its slope at the point for which $x = 1$; $x = 2$; $x = 3$.

Observe that the slope changes from negative on the left of $x = 2$ to positive on the right of $x = 2$.

15. In Ex. 14 find the equation of the tangent at the point for which $x = 2$.

16. Draw the graph of the equation $y = 3x^2 - x^3$, find the slope at the points for which x has the values 1, 2, and 3, and state at what point the curve has a zero slope.

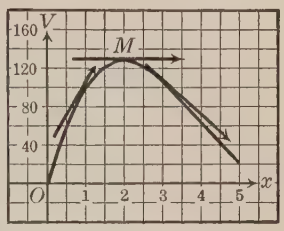
17. Draw the graph of the equation $d = 16t^3 - t^4$ for the interval from $t = 0$ to $t = 16$, and find the slope of the curve at the points for which t has the values 11, 12, and 13 respectively. For which of these values is d the greatest? Show that the slope of the curve is positive when $t < 12$ and negative when $t > 12$.

18. Find the slope of the ellipse $9x^2 + 16y^2 = 144$ at the points for which $x = \pm 2$, and write the equations of the tangents at these points.

5. Derivatives in Relation to Maxima and Minima

Maximum Value and Zero Slope. In the problem on page 239, where the volume V of the box in terms of the side x of the squares to be cut from the corners was given by the equation

$$V = 4x^3 - 48x^2 + 144x,$$



we found from the graph that V is a maximum at the point M, for which $x = 2$. If, as here shown, we now consider the tangents to the curve in the vicinity of M, we see that

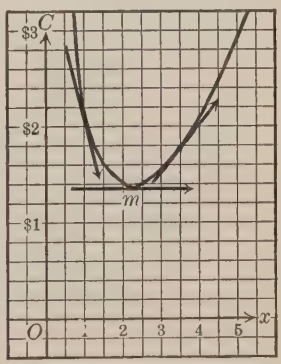
The slope of the tangent is positive to the left of M and negative to the right of M, while at M the tangent is parallel to the x axis and has a zero slope.

That is, at the left of M we have y increasing as x increases, but at the right of M we have y decreasing as x increases.

Minimum Value and Zero Slope. On page 240 we found that in the equation

$$C = 10x^2 + \frac{200}{x},$$

the cost C is a minimum at m, for which x, the side of the base, has the approximate value 2.15. We now see, as above, that



The slope of the tangent is negative to the left of m and positive to the right, while at m the tangent is parallel to the x axis and has a zero slope.

That is, the slope of the tangent is zero for both a maximum or a minimum value, and we can determine which it is by considering the slope of the tangents at the right or left of the maximum or minimum points.

Conditions for Maxima and Minima. Since the tangent at a point on a curve for which the ordinate is either a maximum or a minimum is parallel to the x axis, we see that

When maximum or minimum values occur, $\frac{dy}{dx} = 0$.

Thus, in $y = \frac{1}{3}x^3 - 2x^2 + 3x + 2$,

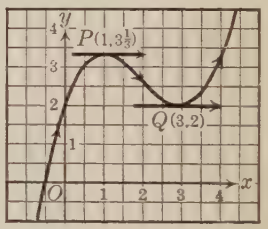
we have $\frac{dy}{dx} = x^2 - 4x + 3$.

Equating this derivative to zero, we then have

$$x^2 - 4x + 3 = 0;$$

whence $x = 1$ or 3.

That is, maximum and minimum values may occur at the points $(1, 3\frac{1}{3})$ and $(3, 2)$.



Designating the point $(1, 3\frac{1}{3})$ as P, we see that just to the left of P, for example, when $x = 0$,

$$\frac{dy}{dx} = 0^2 - 4 \cdot 0 + 3 = 3.$$

Similarly, just to the right of P, for example, when $x = 2$,

$$\frac{dy}{dx} = 2^2 - 4 \cdot 2 + 3 = -1.$$

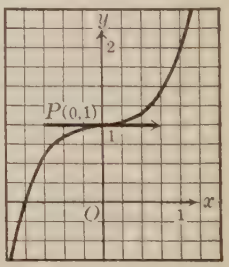
That is, the point $P(1, 3\frac{1}{3})$ is a maximum, and in the same way it can be shown that $Q(3, 2)$ is a minimum.

It is important to observe that while $\frac{dy}{dx} = 0$ is a necessary condition for maxima and minima, it is not a sufficient condition. For example, in the case of

$$y = x^3 + 1$$

we have $\frac{dy}{dx} = 3x^2$.

Hence if $x = 0$, we have $\frac{dy}{dx} = 0$, but, as shown in the graph, the point $P(0, 1)$ is neither a maximum nor a minimum.



Summary. The following outline summarizes the steps to be taken in testing a function for maxima and minima:

1. Find $\frac{dy}{dx}$.

2. Equate this derivative to zero and solve for x, thus finding the values of x for which the graph of the function may have maximum or minimum points.

3. Find the value of $\frac{dy}{dx}$ for values of x which correspond to points in that vicinity on both the right and the left of the points found above, remembering that

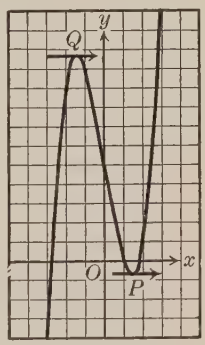
If the derivative is positive on the left and negative on the right, the ordinate is a maximum;

If the derivative is negative on the left and positive on the right, the ordinate is a minimum.

For example, in testing the function $y = x^3 - 6x + 5$ for maximum and minimum values, we may proceed, as follows:

1. $\frac{dy}{dx} = 3x^2 - 6$.

2. From $3x^2 - 6 = 0$ we have $x = \pm\sqrt{2}$; that is, the values of x for which y may be a maximum or a minimum are $\pm\sqrt{2}$.



3. Taking the point for which $x = \sqrt{2}$, if $x = 1$, $\frac{dy}{dx} = -3$; and if $x = 2$, $\frac{dy}{dx} = 6$. That is, y is a minimum when $x = \sqrt{2}$, as shown by the point $P(\sqrt{2}, 5 - 4\sqrt{2})$ in the graph at the right.

Then taking the point for which $x = -\sqrt{2}$, if $x = -2$, $\frac{dy}{dx} = 6$; and if $x = -1$, $\frac{dy}{dx} = -3$. That is, y is a maximum when $x = -\sqrt{2}$, as shown by the point $Q(-\sqrt{2}, 5 + 4\sqrt{2})$.

By the method used on page 241, we could find in this case only approximate values of x for which y is a maximum or a minimum.

Exercises. Maxima and Minima

1. Find the slope of the curve $y = x^2 - 2x + 2$ at the point $P(2, 2)$, and draw the tangent at this point.

2. In Ex. 1 find the slope at the point $Q(1, 1)$, and draw the tangent at this point. Determine whether y has a maximum or a minimum value at Q, and find this value.

3. Find the slope of the curve $y = x^5$ at the points $P(1, 1)$ and $Q(-1, -1)$. State why the function has neither a maximum nor a minimum value at the point where the curve crosses the y axis.

Test each of the following functions for maximum and minimum values, giving the maximum or minimum value in each case:

4. $y = x^3 - x^2$.

5. $y = x^2 + \frac{16}{x}$.

6. $y = -x^2 + 3x + 5$.

7. $y = 2x^2 - 5x + 3$.

8. $y = x^2 + \frac{2}{x} + 3$.

9. $y = 2x^3 - 9x^2 - 24x - 2$.

10. Check the graphic solution of the problem about the volume of the box given on page 239 by finding the derivative of the equation $V = 4x^3 - 48x^2 + 144x$.

11. Check the graphic solution of the problem on page 240 by finding the derivative of the equation $C = 10x^2 + \frac{200}{x}$.

12. In the equation $s = 145.5w^2 + 120w$ find the value of w that will make s a minimum.

13. In the equation $d = 3.2t^2 + 2t + 2$ find the value of t that will make d a maximum or a minimum, and determine which it is.

14. When the speed v, in feet per second, of a moving body is given by the equation $v = 36t^2 - 2t^3$, for what value of t, the time in seconds, is the speed the greatest? After how many seconds is the acceleration the greatest?

15. The waste in a certain type of electric conductor is expressed, in terms of the resistance r, by the formula $w = c^2 r + \frac{k^2}{r}$, where c and k are constants. Find the value of r for which there will be the least waste.

16. When two sources of heat are 16 ft. apart the total intensity (I) of heat at a point which is x feet from the first source is given by the formula $I = \frac{27}{x^2} + \frac{64}{(16 - x)^2}$. Find the value of x for which the intensity will be a minimum.

17. In the equation $I = \frac{I_1}{x^2} + \frac{I_2}{(d - x)^2}$, where I_1, I_2, and d are constants, show that I is least when $\frac{d - x}{x} = \sqrt[3]{\frac{I_2}{I_1}}$.

18. If a ball is thrown vertically upward with an initial velocity of 72 ft./sec., its height after t seconds is given by the formula $h = 72\,t - 16\,t^2$. At what time does the ball reach its maximum height?

19. The relation between the height y and the horizontal distance x reached by a projectile at various points in its flight is given by the formula $y = x \tan \alpha - \frac{g x^2}{2\,v_0^2 \cos^2 \alpha}$, where α is the angle at which it is fired, g has the approximate value 32, and v_0 is the initial velocity. Taking $\alpha = 45°$ and $v_0 = 2000$ ft./sec., find the maximum height reached.

This formula disregards the resistance offered by the air.

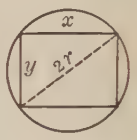
20. In building a closed tank with a square base that shall have a capacity of 500 cu. ft., what dimensions should be allowed in order to use the least amount of material?

21. In drawing up the specifications for an order of cylindric tin cans that are to have a capacity of 1 gal. (231 cu. in.), what radius of base should be specified in order to require the least amount of tin?

Disregard any question of waste in cutting the round bases.

22. A rectangular inclosure containing 800 sq. yd. is to be fenced off along a stone wall, which will serve as the fence on one side. Find the dimensions that should be planned for the inclosure in order to require the least amount of fencing.

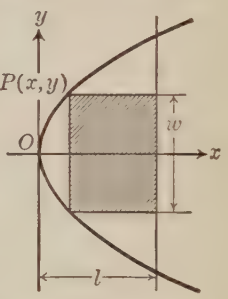
23. Find the dimensions of the maximum rectangle that can be inscribed in a circle which has the radius r.



"Maximum" in this case refers, of course, to area.

24. The lower corner of a page of a book which is 5 in. wide is folded over so as just to reach the inside edge of the page. Find the length of the part of the lower edge folded over when the length of the crease is the shortest.

25. Find the width w of the maximum rectangle that can be inscribed in the segment of the parabola $y^2 = 4px$ cut off by a line parallel to the y axis and l units from the vertex.



26. Find a number which, added to its reciprocal, will give a minimum sum.

27. What dimensions should be used for a rectangular inclosure containing 10,000 sq. yd. in order to require the least amount of fencing?

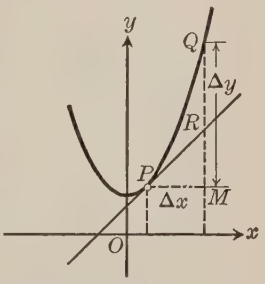
28. A closed rectangular box with a square base is to have a capacity of 72 cu. ft. If the material for the top and bottom can be bought at 4¢/sq. ft., and that for the sides at 2¢/ sq. ft., what dimensions will make the cost of material the least?

29. The ends of a closed cylindric tank which is to have a capacity of 1500 cu. ft. are to be constructed of heavy material costing 80¢/sq. ft., while the curve surface is to be made of lighter material costing 60¢/sq. ft. Find the radius that will make the cost of material for the tank a minimum.

6. Differentials

Differentials. In certain applications of the calculus it is convenient to use what is known as *differential notation* and to speak of *differentials*. A rigorous definition of a differential requires certain theorems with respect to infinitesimals that are beyond the scope of our present work, but it is possible to give a brief statement of the general ideas that are involved.

In our earlier work the symbol $\frac{dy}{dx}$ was defined as the limit of $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$. In the accompanying figure, in which $\frac{\Delta y}{\Delta x} = \frac{MQ}{PM}$, we see that geometrically $\frac{dy}{dx} = \frac{MR}{PM}$, which is the slope of the tangent at P.



Let us now consider dy and dx as two quantities such that $dy \div dx = \frac{MR}{PM}$, or $dy = \frac{MR}{PM} \cdot dx$. It then follows that

$$dy = \frac{dy}{dx} \cdot dx.$$

We call the quantities dx and dy the *differentials of x and y* respectively. That is,

The differential (dy) of the dependent variable is the product of the derivative of the function and the differential of the independent variable.

For example, when $y = x^2$, we have $\frac{dy}{dx} = 2x$, and hence, by the above definition, $dy = 2x\,dx$.

In the above figure, as already stated, we have

$$\frac{\Delta y}{\Delta x} = \frac{MQ}{PM} \quad \text{and} \quad \frac{dy}{dx} = \frac{MR}{PM};$$

that is, for the differential of x we have $dx = \Delta x$, but for the differential of y we have $dy = MR$, while $\Delta y = MQ$.

Finding a Differential. In finding the differential dy in the function

$$y = 3x^3 - 6x^2 + 7x + 5,$$

we first find the derivative; that is, we find that

$$\frac{dy}{dx} = 9x^2 - 12x + 7.$$

Then, by the definition given on page 287, we have

$$dy = (9x^2 - 12x + 7)dx.$$

Exercises. Finding Differentials

Find the differential of the dependent variable in each of the following functions:

1. $y = x^2 - 0.3x^3$.
2. $y = 2x(x^2 - 1)$.
3. $d = 240t - 16t^2$.
4. $P = 2c - 0.01c^2$.
5. $v = 5 + 3t + 2t^2$.
6. $d = 16t^2 + 100t$.
7. $d = 1.6t^2 + 2t + 10$.
8. $y = x(100 - x)$.
9. $y = 2x^2 - 3x + 1$.
10. $y = \frac{1}{2}x^3 - 3x^2 - 2$.
11. $y = 0.2x^4 - 0.03x^2$.
12. $y = 1.3x^3 + 2.5x$.
13. $y = 3.4x^5 - 2.25x^4$.
14. $C = 50x^2 + \dfrac{800}{x}$.
15. $y = \dfrac{1+x}{1-x}$.
16. $y = \dfrac{x^2 + 2x + 1}{x^2 - 2x + 1}$.
17. $A = x^2$.
18. $V = x^3$.
19. $A = \pi r^2$.
20. $V = \frac{4}{3}\pi r^3$.
21. $V = 400P^{-1}$.
22. $V = 5x^3$.
23. $y = 5 + 2x + 3x^2$.
24. $y = 15x(15 + x)$.
25. $y = x^3 + 3x^2 + 3x$.
26. $y = x^3 - 3x - 1$.
27. $y = 0.7x^5 - 0.9x^4$.
28. $y = 0.05x^4 + 0.04x^5$.
29. $y = x^5 + x^4 - x^2 - 1$.
30. $y = \dfrac{x}{x^2 + 1}$.
31. $y = \dfrac{x^2 + 1}{x^2 - 1}$.
32. $y = \dfrac{2x^2 - 3x + 2}{2x^2 + 3x + 2}$.

Approximate Change in a Function. If we let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be any two points on the graph of the function $y = \frac{1}{2}x^2$, we have

$$y + \Delta y = \tfrac{1}{2}(x + \Delta x)^2 = \tfrac{1}{2}x^2 + x\,\Delta x + \tfrac{1}{2}\overline{\Delta x}^2.$$

Subtracting $y = \frac{1}{2}x^2$,

we then have $\Delta y = x\,\Delta x + \frac{1}{2}\overline{\Delta x}^2$,

which is the exact change in y when x changes by the amount Δx.

Taking the derivative of $y = \frac{1}{2}x^2$,

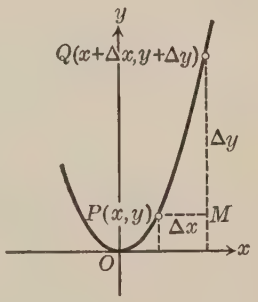
we then have $\dfrac{dy}{dx} = x$;

whence $dy = x\,dx$.

Then, since $dx = \Delta x$,

we see that $\Delta y = x\,dx + \frac{1}{2}\overline{\Delta x}^2$

$= dy + \frac{1}{2}\overline{\Delta x}^2$.



That is, Δy is greater than dy by $\frac{1}{2}\overline{\Delta x}^2$. But if Δx is a small fraction, $\frac{1}{2}\overline{\Delta x}^2$ is much smaller, and hence we may take dy as an approximate value of Δy, the exact change in y.

If in the above case we let $x = 10$ and $\Delta x = 0.1$, we have

$$\Delta y = x\,\Delta x + \tfrac{1}{2}\overline{\Delta x}^2 = 1 + 0.005 = 1.005,$$

and $dy = x\,dx = x\,\Delta x = 1$;

that is, dy is a fairly close approximation to the exact change 1.005.

As another example, suppose that we wish to find the approximate change produced in the area A of a square by changing the length of its side x from 5 in. to 5.02 in.

We have $A = x^2$,

from which $dA = 2\,x\,dx$.

Then, since $x = 5$ and $dx = 0.02$, we have

$$dA = 2 \times 5 \times 0.02 = 0.2;$$

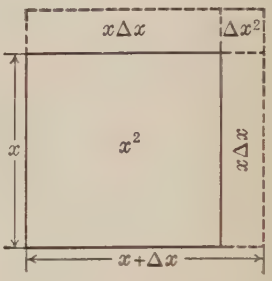
that is, the approximate change in the area of the square is 0.2 sq. in. The precise change is $5.02^2 - 5^2$, or 0.2004 sq. in., but it is evident that for many purposes the approximation will be sufficiently close.

Percentage Error. If in measuring a square in which the true length of the side is x we take the measured value of the side as $x + \Delta x$, we see that for the true area A we have

$$A = x^2,$$

and for the measured area

$$A + \Delta A = x^2 + 2\,x\,\Delta x + \overline{\Delta x}^2.$$



The exact error ΔA is therefore given by

$$\Delta A = 2\,x\,\Delta x + \overline{\Delta x}^2,$$

and the approximate error dA by

$$dA = 2\,x\,dx.$$

The difference between these errors is $\overline{\Delta x}^2$, the area of the small square in the corner, which we may disregard.

The ratio, expressed as a per cent, of error to true value is called the *percentage error*.

For example, if we make an error of 0.5 sq. in. in an area which has a true value of 25 sq. in., the percentage error is $0.5 \div 25$, or 2 %.

In the above case the exact percentage error is $\frac{\Delta A}{A}$, but we usually take the approximate percentage error $\frac{dA}{A}$.

For example, find the approximate error and also the percentage error in computing the volume of a cube from a measured length of 5 in. when the true length is 4.97 in.

From $$V = x^3,$$

we have $$dV = 3\,x^2\,dx,$$

where $x = 4.97$ and $dx = \Delta x = 5 - 4.97 = 0.03$. Using logarithms, we then find that

$$dV = 3 \times 4.97^2 \times 0.03 = 2.223;$$

that is, the approximate error in computing the volume is 2.223 cu. in.

The percentage error is therefore $2.223 \div 4.97^3$. Using logarithms again, we then find this error to be 0.01810, or 1.81%.

A simpler way of obtaining an approximate result is to let $x = 5$. We then have

$$dV = 3 \times 5^2 \times 0.03 = 2.25,$$

and $$\frac{dV}{V} = 2.25 \div 125 = 0.018 = 1.8\%.$$

Exercises. Approximate Changes

1. In the function $y = 8x^2 - 0.2x^3$ find the approximate change in y when x increases from 3 to 3.03.

2. Find the approximate change in P when c changes from 10 to 10.003 in the formula $P = 2c - 0.01c^2$.

3. In a square which has an area of 25 sq. in. find the approximate change in the length of the side that will produce a change of 0.8 sq. in. in the area.

In Exs. 4–8 find the approximate change in

4. The area of a circle when the radius changes from 6 to 6.1; from 6 to 6.05.

5. The volume of a cube when the edge changes from 10 to 9.99; from 10 to 9.97.

6. The area of the total surface of the cube in Ex. 5.

7. The surface area of a sphere when the radius changes from 3 to 3.01.

8. The volume of the sphere in Ex. 7.

9. If heating a circular metal plate causes the radius to change from 5 in. to $5\frac{1}{16}$ in., what is the approximate change in the area of the plate?

10. For a certain gas the formula $V = \frac{600}{P}$ expresses the relation between the volume V and the pressure P. If the pressure changes from 400 to 400.2, what is the approximate change in volume?

11. If in Ex. 10 the pressure is read as 499.6 when it is really 500, what is the approximate error in the calculated volume? the approximate percentage error?

12. In laying out a square which is to have an area of 16 sq. in., what error may be allowed in measuring the side if the error in the area is not to be greater than 0.1 sq. in.?

13. If the length of the edge of a cube is measured and found to be 15 in., when it is really 14.98 in., what is the approximate error in using the measured value to compute the volume? to compute the area of the total surface? Find the percentage error in each case.

14. Taking x as the length of the edge of a cube and V as its volume, find ΔV and dV when the length of the edge changes by Δx, and draw a figure to illustrate the difference between ΔV and dV.

15. Taking h as the height of a right circular cylinder with the radius r, show that the approximate change (dV) in the volume when r is increased by Δr is given by the formula $dV = 2\,\pi rh\,\Delta r$.

16. If a right circular cylinder 10 in. high has its radius increased from 5 in. to 5.02 in., what is the approximate change in the volume?

17. The formula for the volume V of a right circular cone in terms of the height h and the radius r is $V = \frac{1}{3}\,\pi r^2 h$. Write a formula for the volume of such a cone when the height is equal to the radius, and from it find the approximate change in volume when r is increased by Δr.

18. Write a formula for the exact volume of the metal in a hollow sphere in which the inside radius is x and the thickness of the metal shell is Δx. Find a formula for the approximate volume of the metal, and apply it in finding the amount of metal in a spherical shell with an inside radius of 8.5 in. and a thickness of 0.08 in.

19. Find the capacity in cubic feet of a box x feet long, x feet wide, and x feet deep, inside measurements. Taking $x = 6$, find the approximate number of cubic feet of lumber $\frac{1}{2}$ in. thick necessary to make a closed box of these dimensions. Solve the problem by using differentials, and check the result by arithmetic.

7. Integration

Differentiation and Integration. As we have seen, the process of differentiation enables us to find the rate of change in a dependent variable with respect to the independent variable when the functional relation that connects the two is known. For example, given the functional relation

$$y = \tfrac{1}{2} x^2,$$

we find that

$$\frac{dy}{dx} = x.$$

Suppose that we know that the rate of change of y with respect to x is x; that is, that we are given

$$\frac{dy}{dx} = x.$$

What is then the functional relation that exists between x and y? It is evident from the above discussion that one answer is

$$y = \tfrac{1}{2} x^2,$$

but since the derivative of a constant with respect to a variable is zero, we may equally well have

$$y = \tfrac{1}{2} x^2 + 1,$$

$$y = \tfrac{1}{2} x^2 + 2,$$

or

$$y = \tfrac{1}{2} x^2 + c,$$

where c represents any constant. This process of finding the functional relation, which is the inverse of differentiation, is called *integration.*

It is evident that integration, unlike differentiation, does not lead to a unique result. That is, in the above case the result of the integration is

$$y = \tfrac{1}{2} x^2 + c,$$

where the constant c may have any value. We know, however, that functions which have the same derivative differ only by a constant, and hence, if we can in some way determine the constant, we can find the functional relation when given its derivative.

Constant of Integration. To sum up the discussion on page 293, we see that if we are given the derivative $\frac{dy}{dx} = x$, we can find by integration that

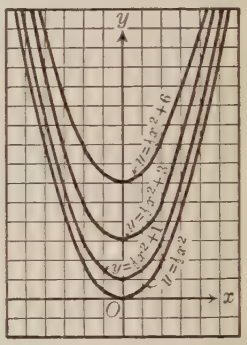
$$y = \tfrac{1}{2}x^2 + c,$$

where c, which represents any constant, is known as the *constant of integration.*

Assigning convenient values to c, we may now make the following table:

If $c =$	0	1	3	6
$y =$	$\frac{1}{2}x^2$	$\frac{1}{2}x^2 + 1$	$\frac{1}{2}x^2 + 3$	$\frac{1}{2}x^2 + 6$

If we then draw the graphs of these functions, as shown in the figure at the right, we have what is known as a *family of curves.*



In a family of curves the equations are always the same except for the arbitrary constants. Such constants are often called *parameters.*

In this particular case we see that the constant c determines the point where the graph of the function cuts the y axis; that is, the curve which is represented by the equation $y = \frac{1}{2}x^2 + c$ is a parabola with the intercept c on the y axis.

Determining the Constant. If in the above case we know that y has a certain value for some specified value of x, it is possible to determine the value of c. For example, if we know that $y = 8$ when $x = 2$, we may substitute these values in the equation $y = \frac{1}{2}x^2 + c$, and find that $8 = 2 + c$, or $c = 6$. We thus find that under these conditions the functional relation between x and y is

$$y = \tfrac{1}{2}x^2 + 6;$$

that is, this is the equation of the parabola for which $\frac{dy}{dx} = x$ and which also passes through the point (2, 8).

Using Integration. We shall now see how integration can be applied in the solution of a problem, taking for this purpose a simple problem about moving bodies such as frequently arises in the study of physics.

For example, suppose that a ball is thrown vertically upward from the ground and that t seconds after it leaves the ground its speed is approximately $96 - 32\,t$ feet per second. Find an expression for the height of the ball at the end of t seconds, and then find this height when $t = 5$.

If we let y represent the height of the ball at the end of t seconds, we see that for the rate of change of y with respect to t (the speed of the ball) we have

$$\frac{dy}{dt} = 96 - 32\,t.$$

We must therefore find some function of t which, when differentiated, gives $96 - 32\,t$. A little inspection shows that this function is of the form

$$y = 96\,t - 16\,t^2 + c.$$

That is, the height y of the ball at the end of t seconds is given by the equation $y = 96\,t - 16\,t^2 + c$. If we can now determine the value of c, we can solve the problem.

By the nature of the problem we know that $y = 0$ when $t = 0$; that is, at 0 sec. the ball has not started upward. Substituting 0 for t and y, we have

$$0 = 96 \cdot 0 - 16 \cdot 0 + c,$$

or

$$c = 0;$$

that is, the height y of the ball is given by the equation $y = 96\,t - 16\,t^2$.

Hence, when $t = 5$, we can easily find that

$$y = 96 \cdot 5 - 16 \cdot 25 = 480 - 400 = 80;$$

that is, at the end of 5 sec. the height reached by the ball is 80 ft.

Continuing the discussion, suppose that we now wish to find how many seconds elapse between the time that the ball is thrown upward and the time that it returns to the ground.

In this case we obviously have $y = 0$; that is, we have

$$0 = 96\,t - 16\,t^2;$$

whence

$$t = 0 \text{ or } 6.$$

That is, the ball will return to the ground in 6 sec., the value 0 for y simply showing the position of the ball before it started.

Integrating Powers of x. Proceeding as on page 293, we may now continue our examination of functional relations which correspond to given derivatives. Thus, if the given derivative involves the power of x shown in the first column of the table at the right, we readily see that the functional relation between x and y is of the form shown in the second column.

$\frac{dy}{dx}$	Functional Relation
x	$y = \frac{1}{2}x^2 + c$
x^2	$y = \frac{1}{3}x^3 + c$
x^3	$y = \frac{1}{4}x^4 + c$
x^4	$y = \frac{1}{5}x^5 + c$

We may therefore infer from this table and from the preceding discussion that if the given derivative is

$$\frac{dy}{dx} = x^n,$$

the functional relation between x and y is

$$y = \frac{x^{n+1}}{n+1} + c.$$

Although this inference is a natural one, and is, in general, correct, it fails to hold when $n = -1$. The reason, of course, is that we then have $n + 1 = 0$, and a zero divisor is not permissible. In all other cases this formula for the functional relation is valid.

For example, if we are given that $\frac{dy}{dx} = \sqrt{x}$, or that $\frac{dy}{dx} = x^{\frac{1}{2}}$, we may apply the above formula and have

$$y = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \tfrac{2}{3}\, x\sqrt{x} + c.$$

Differential Forms. The same law is readily applied in case the differential of y is given. For example, if given that

$$dy = x^2\, dx,$$

we have

$$\frac{dy}{dx} = x^2.$$

Hence, as above,

$$y = \frac{x^{n+1}}{n+1} + c$$

$$= \tfrac{1}{3}x^3 + c.$$

In the same way, if $dy = x^5\, dx$, we have $y = \frac{1}{6}x^6 + c$.

Exercises. Integration

Using the method given on page 296, find the value of y for each of the following derivatives:

1. $\frac{dy}{dx} = x^4$.
2. $\frac{dy}{dx} = x^{10}$.
3. $\frac{dy}{dx} = x^{\frac{1}{3}}$.
4. $\frac{dy}{dx} = \sqrt[4]{x}$.
5. $\frac{dy}{dx} = \sqrt[3]{x^2}$.
6. $\frac{dy}{dx} = x^{\frac{5}{2}}$.
7. $\frac{dy}{dx} = x^{-2}$.
8. $\frac{dy}{dx} = \frac{1}{x^3}$.
9. $\frac{dy}{dx} = \frac{1}{\sqrt{x^3}}$.

10. If in Exs. 1–9, above, $y = 3$ when $x = 1$, what is the value of the constant of integration in each case?

Find the functional relation that exists between x and y when

11. $dy = x^6\,dx$.
12. $dy = x^{\frac{1}{5}}\,dx$.
13. $dy = x^{\frac{3}{2}}\,dx$.
14. $dy = \sqrt[5]{x^2}\,dx$.
15. $dy = \sqrt{x^5}\,dx$.
16. $dy = x^{-2}\,dx$.
17. $dy = x^{-\frac{1}{2}}\,dx$.
18. $dy = x^{-3}\,dx$.

19. If the speed of a particle t seconds after starting from rest is $40\,t - 3\,t^2$ feet per second, what is an expression for the distance traveled by the particle in t seconds? Find this distance when $t = 5$; when $t = 10$.

That is, as on page 295, first find the function for which $\frac{dy}{dt} = 40\,t - 3\,t^2$.

20. Find the equation of the curve which passes through the point (3, 1) and has the slope $3\,x^2$.

21. Find the equation of the curve which passes through the origin and whose slope is k times the abscissa of any point. Plot the equation for the values 1, 2, 3, and 4 for k.

22. If the slope of a curve at the point $P(1, 1)$ is $\frac{k}{x^2}$, what is the equation of the curve? Plot the equation for the values 1, 2, 3, and 4 for k.

Integral Sign. We have seen that if the derivative of a function is known, the differential is also known. That is, if we are given that

$$\frac{dy}{dx} = x,$$

we may write

$$dy = x\,dx.$$

In connection with integration we therefore think of y as an expression in x which has a certain differential.

The operation of integration is indicated by the symbol $\int$, which is known as the *integral sign.*

The symbol is an old form of letter *s*, which is the initial letter of the Latin word *summa*, meaning a sum.

The integral sign is written before the differential of the function; that is, if we are given that

$$dy = x\,dx$$

we may first write

$$y = \int x\,dx,$$

and then

$$y = \tfrac{1}{2}x^2 + c.$$

Summary of Integration. Just as on page 271 we summarized the various differentiation formulas, so we shall now summarize what we have learned about integration. Corresponding to the two formulas numbered 6 on page 271, we have

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c,$$

and

$$\int kx^n dx = \frac{kx^{n+1}}{n+1} + c,$$

and, corresponding to the formula numbered 3, we have

$$\int (x^n + x^m) dx = \frac{x^{n+1}}{n+1} + \frac{x^{m+1}}{m+1} + c.$$

The second and third formulas, above, are easily developed by the method used on page 296 for the first one. These formulas will be more convenient than finding the function by inspection, as was done in the problem on page 295.

Using the Formulas. The integration formulas on page 298 are easily applied; for example, given that

$$\frac{dy}{dx} = 4x^3 - x^2 + 1$$

and that $y = 2$ when $x = 1$, express y as a function of x.

From $\frac{dy}{dx} = 4x^3 - x^2 + 1,$

we have $dy = (4x^3 - x^2 + 1)dx;$

whence $y = \int (4x^3 - x^2 + 1)dx$

$= x^4 - \frac{1}{3}x^3 + x + c.$

Substituting 1 for x and 2 for y, we then have

$2 = 1 - \frac{1}{3} + 1 + c,$

or $c = \frac{1}{3},$

and hence $y = x^4 - \frac{1}{3}x^3 + x + \frac{1}{3}.$

Exercises. Integration

Express y as a function of x, given that

1. $\frac{dy}{dx} = 2x^2 - x + 5$, and $y = 4$ when $x = 2$.

2. $\frac{dy}{dx} = \frac{3}{x^2} + 6$, and $y = 10$ when $x = 1$.

3. $\frac{dy}{dx} = 3x^2 - 4x^3$, and $y = 0$ when $x = 0$.

4. $\frac{dy}{dx} = ax + b$, and $y = 2$ when $x = 0$.

5. $\frac{dy}{dx} = ax^2 + bx + c$, and $y = a$ when $x = 0$.

6. $\frac{dy}{dx} = \frac{\sqrt[3]{x} - \sqrt{x} + 2x}{x}$, and $y = 5$ when $x = 1$.

7. $\frac{dy}{dx} = \frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{3}{2}}}$, and $y = 3$ when $x = 4$.

8. $\frac{dy}{dx} = \frac{x^{\frac{2}{3}} + x^{\frac{1}{3}} + 3}{x^{\frac{2}{3}}}$, and $y = 2$ when $x = 1$.

Perform each of the following integrations:

9. $\int x^7\,dx.$
10. $\int x^{2a}\,dx.$
11. $\int x^{2a+1}\,dx.$
12. $\int x^{\frac{2}{3}}\,dx.$
13. $\int \sqrt[5]{x^4}\,dx.$
14. $\int x^{-a}\,dx.$
15. $\int x^{-\frac{1}{2}}\,dx.$
16. $\int x^{-\frac{2}{3}}\,dx.$
17. $\int 5\,x^4\,dx.$
18. $\int -2\,x^{-2}\,dx.$
19. $\int x\sqrt{x}\,dx.$
20. $\int x^2\sqrt{x}\,dx.$

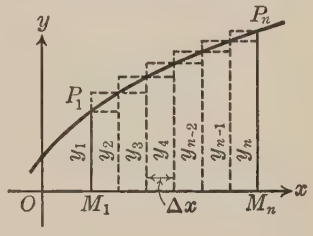
21. If a body that starts from rest acquires a speed of $4.5\,t^2$ feet per second at the end of t seconds, in how many seconds will it be 1500 ft. from the starting-point?

Perform each of the following integrations:

22. $\int (2\,t-2)dt.$
23. $\int (960-32\,t)dt.$
24. $\int (2\,ax+b)dx.$
25. $\int (x^{\frac{1}{2}}-x^{-\frac{1}{2}}+2)dx.$
26. $\int (x^2+3\,x)dx.$
27. $\int (3\,x^2-0.2\,x)dx.$
28. $\int (3\,x^2+2\,x+5)dx.$
29. $\int \frac{75\sqrt[3]{x}}{x^2}\,dx.$
30. $\int -5\cdot\frac{1}{\sqrt{x^3}}\,dx.$
31. $\int -\frac{2}{3}\cdot\frac{1}{x^{\frac{3}{2}}}\,dx.$
32. $\int (0.3\,x^4-0.002\,x)dx.$
33. $\int (x^{\frac{5}{2}}+x^{\frac{3}{2}}-x^{\frac{1}{2}}+6)dx.$
34. $\int (20\,t^2-4\,t+2)dt.$
35. $\int (3\theta^2-2\theta+1)d\theta.$
36. $\int (2\,x^3-3\,x^2+4\,x+5)dx.$
37. $\int (2\,t^2-0.3\,t^3-0.1\,t^4)dt.$
38. $\int (3\,t^{-3}+t^{-2}+5)dt.$
39. $\int (0.005\,x^2-0.01\,x^4)dx.$
40. $\int \left(x^2+\frac{2}{x^2}+5\right)dx.$
41. $\int \frac{x^3+2\,x^2+6}{x^2}\,dx.$

Rectangle Method of Areas. Probably the most familiar case of a variable approaching a limit is that of the area of a circumscribed or inscribed regular polygon approaching the area of the circle as its limit when the number of sides is increased indefinitely.

In the same way, in this figure we may regard the area of the surface which is bounded by the smooth curve between the points P_1 and P_n, the ordinates M_1P_1 and M_nP_n, and the x axis either as the limit of the sum of certain "inscribed rectangles" or as the limit of the sum of the corresponding "circumscribed rectangles."



As shown by the dotted lines in the figure, we may first divide M_1M_n into n equal parts Δx, and we may then construct the inscribed rectangles in which the altitudes are $y_1, y_2, \cdots, y_{n-2}, y_{n-1}$ and the circumscribed rectangles in which the altitudes are $y_2, y_3, \cdots, y_{n-1}, y_n$.

Then, since Δx is the base in each set of rectangles, we see that if we let A_I represent the sum of the areas of the inscribed rectangles and A_C the sum of the areas of the circumscribed rectangles, we have

$$A_I = y_1\,\Delta x + y_2\,\Delta x + y_3\,\Delta x + \cdots + y_{n-1}\,\Delta x,$$

and $$A_C = y_2\,\Delta x + y_3\,\Delta x + y_4\,\Delta x + \cdots + y_n\,\Delta x.$$

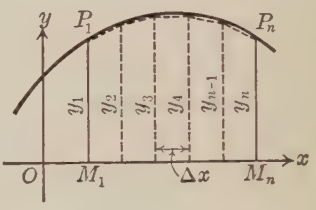
It is evident that either A_I or A_C is a rough approximation to the area under the curve between M_1P_1 and M_nP_n. Furthermore, if we decrease the interval Δx, the approximation will become closer. That is, letting A represent the area of the surface bounded as shown, we have

$$A = \lim_{n \to \infty} (y_1\,\Delta x + y_2\,\Delta x + y_3\,\Delta x + \cdots + y_{n-1}\,\Delta x),$$

or $$A = \lim_{n \to \infty} (y_2\,\Delta x + y_3\,\Delta x + y_4\,\Delta x + \cdots + y_n\,\Delta x).$$

Trapezoidal Formula. Instead of using rectangles, as on page 301, a closer approximation to the area under a curve can be obtained by the use of trapezoids.

If in this figure we represent the area of the first trapezoid by T_1, that of the second by T_2, and so on, we have



$$T_1 = \tfrac{1}{2}(y_1 + y_2)\Delta x,$$
$$T_2 = \tfrac{1}{2}(y_2 + y_3)\Delta x,$$

and so on to the last, for which

$$T_n = \tfrac{1}{2}(y_{n-1} + y_n)\Delta x.$$

Letting A represent the sum of the areas of the trapezoids, we then have

$$A = \tfrac{1}{2}[(y_1 + y_2) + (y_2 + y_3) + (y_3 + y_4) + \cdots + (y_{n-1} + y_n)]\Delta x$$
$$= \tfrac{1}{2}[(y_1 + y_n) + 2(y_2 + y_3 + y_4 + \cdots + y_{n-1})]\Delta x,$$

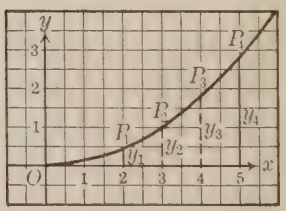
or $$A = \left[\tfrac{1}{2}(y_1 + y_n) + y_2 + y_3 + y_4 + \cdots + y_{n-1}\right]\Delta x.$$

In this formula, which is known as the *Trapezoidal Formula* for areas, it is evident that the sum of the areas of the trapezoids approaches the area under the curve when $n \to \infty$.

Using the Formula. The following examples illustrate the use of the Trapezoidal Formula:

1. Find the approximate area under the curve $y = \frac{1}{9}x^2$ between the ordinates at the points for which $x = 2$ and $x = 5$.

By locating points in the usual way we first draw the graph of the given equation, as shown at the right.



Taking $\Delta x = 1$, as shown by the figure, we see that when $x = 2$, $y_1 = \frac{1}{9} \cdot 2^2 = 0.44$; when $x = 3$, $y_2 = \frac{1}{9} \cdot 3^2 = 1.00$; when $x = 4$, $y_3 = \frac{1}{9} \cdot 4^2 = 1.78$; and finally when $x = 5$, $y_4 = \frac{1}{9} \cdot 5^2 = 2.78$.

Substituting these values in the Trapezoidal Formula, we then have

$$A = [\tfrac{1}{2}(0.44 + 2.78) + 1.00 + 1.78] \cdot 1 = 4.39;$$

that is, the required area is approximately 4.39 square units.

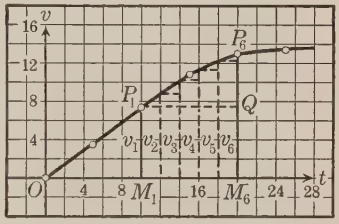
2. In a series of experiments with a car starting from rest the speed v, in feet per second, was found to be related to the time t, in seconds, as shown by the following table:

If $t=$	0	5	10	15	20	25
$v=$	0	3.7	7.5	10.9	13.0	13.7

Find the approximate distance traveled by the car from the tenth to the twentieth second.

From the above table we first draw the accompanying graph showing the speed v as a function of the time t.

In a case involving a uniform speed v we should have $d = vt$, where d is the distance traveled. Thus, if v in the above case had been uniform between $t=10$ and $t=20$, the distance traveled (d) would be vt; that is, it would be represented by the area of the rectangle $P_1M_1M_6Q$ in the figure.



Although the speed is not constant, it is evident that by dividing M_1M_6 into equal parts, the approximate distance traveled by the car between the tenth and twentieth seconds is represented by the sum of the areas of the inscribed rectangles shown in the figure. The exact distance traveled is the area under the curve between the ordinates at the points for which $t = 10$ and $t = 20$.

We may therefore use the Trapezoidal Formula to find an approximation to this area. Estimating the ordinates at the points for which t has the values 12, 14, 16, and 18, we have $y_1 = v_1 = 7.5$, $y_2 = v_2 = 9.0$, $y_3 = v_3 = 10.3$, $y_4 = v_4 = 11.4$, $y_{n-1} = v_5 = 12.3$, $y_n = v_6 = 13.0$, and $\Delta x = 2$. We therefore have

$$A = [\tfrac{1}{2}(7.5 + 13.0) + 9.0 + 10.3 + 11.4 + 12.3] \cdot 2 = 106.5.$$

That is, the approximate distance traveled by the car between the tenth and twentieth seconds is 106.5 ft.

If we should solve this problem by the Rectangle Method described on page 301, we should have

$$A_I = 2\,(7.5 + 9.0 + 10.3 + 11.4 + 12.3) = 101.0,$$

and

$$A_C = 2\,(9.0 + 10.3 + 11.4 + 12.3 + 13.0) = 112.0.$$

The arithmetic mean between these two, which is the approximate distance, is then 106.5 ft., as above.

Work. If a constant force F moves a body a distance x in the direction of the force, the product of the force and the distance is called the *work* done; that is, representing the work by W, we have

$$W = Fx.$$

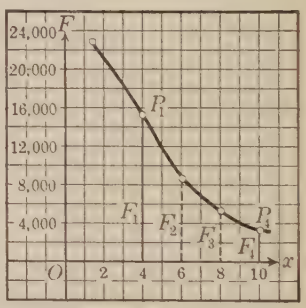
If the force is not constant, the formula does not hold; but just as on page 303 we found an approximate value for the distance traveled by a body moving at a varying speed, so by the same method we can find an approximate value for the amount of work done when the force varies with the distance.

For example, in an engine it is found that the total pressure (force) F, in pounds, on the piston is related to the movement x, measured in inches, of the piston from the start of the stroke as shown by the following table:

If $x =$	1.4	4	6	8	10
$F =$	23,000	15,500	8400	5500	3700

Find the approximate work done while the piston is moving from the point for which $x = 4$ to the point for which $x = 10$.

We first draw the accompanying graph by locating points from the table of values given above. Then, since $W = Fx$, the approximate amount of work done can be found by computing the area under the curve between the ordinates at the points for which $x = 4$ and $x = 10$.



Using the Trapezoidal Formula, we have $y_1 = F_1 = 15{,}500$, $y_2 = F_2 = 8400$, $y_{n-1} = F_3 = 5500$, $y_n = F_4 = 3700$, and $\Delta x = 2$. We therefore have

$$A = [\tfrac{1}{2}(15{,}500 + 3700) + 8400 + 5500] \cdot 2 = 47{,}000.$$

Since the distance x is measured in inches, the result in this case is 47,000 inch-pounds. The unit of work in more common use is the foot-pound (ft.-lb.), and hence for the amount of work done in this case we have $\frac{1}{12} \times 47{,}000$, or $3916\frac{2}{3}$ ft.-lb.

Exercises. Trapezoidal Formula

1. Compute the ordinates of the curve $y = \frac{1}{2}x^2$ at the points for which x has the values 1, 2, 3, 4, and 5. By means of the Trapezoidal Formula find the approximate area under the curve between the points for which $x = 1$ and $x = 5$, and determine whether this is greater or less than the exact area.

2. Compute the ordinates of the curve $y = 0.2\,x^2$ at the points for which x has the values 0, 1, 2, 3, $\cdots$, 10, and find the approximate area under the curve from $x = 0$ to $x = 10$.

3. If the relation between the speed v of an object, in feet per second, and the time t, in seconds, is expressed by the formula $v = 2\,t^2 - 0.4\,t$, what is the distance traveled by the object in the interval from $t = 2$ to $t = 4$?

4. The following table shows the relation between the speed v of an object in feet per second and the time t in seconds:

If $t =$	0	2	2.5	3	3.5	4
$v =$	0	80	106	139	178	225

Draw a graph of the curve and compute the distance traveled in the interval from $t = 0$ to $t = 4$.

5. The relation between a certain force of F pounds and the distance x, in feet, through which it acts is as follows:

If $x =$	4	5	6	8	10
$F =$	71	49	38	31	28

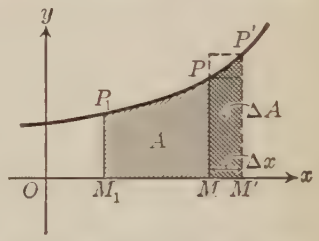
Draw a graph and compute the work done by the force in the interval from $x = 4$ to $x = 10$.

6. The relation between a certain force of F pounds and the distance x, in feet, through which it acts is given by the formula $F = 10\,x^{-1}$. Draw a graph and compute the work done by the force in the interval from $x = 1$ to $x = 5$.

Area under a Curve. As we have seen, the Trapezoidal Formula enables us to find a fairly close approximation to the area under a curve. We shall now consider another method in which we use integration to find the precise area bounded by the curve, two given ordinates, and the x axis. In this method it is necessary to know the equation represented by the curve, which is assumed to be continuous in the interval between the given ordinates.

The Trapezoidal Formula can, of course, be used when the equation of the curve is known, as in the illustrative example on page 302. The method that we shall now learn offers a better way of handling such cases, and hence the chief use of the Trapezoidal Formula is in cases where the equation of the curve is not known, as in the illustrative examples on pages 303 and 304.

Area Increment. In this figure let A be the area bounded by the given curve (whose equation is known), the fixed ordinate M_1P_1, the variable ordinate MP, and the x axis.



When x takes an increment Δx, the variable ordinate MP taking some such position as $M'P'$, it is evident that A is increased by the area under the curve between the points P and P'. This added area is called the *area increment* and is designated by the symbol ΔA.

If we now construct the inscribed and circumscribed rectangles of the area increment, as shown in the figure, and designate the area of the inscribed rectangle as R_I and the area of the circumscribed rectangle as R_C, we see that

$$R_I < \Delta A < R_C.$$

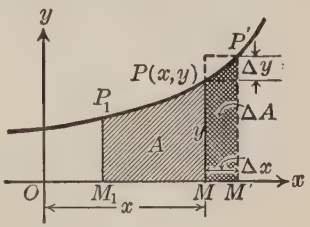
It makes no difference whether the curve runs downward instead of upward as x increases, as the student can easily verify by drawing a figure. That is, ΔA is always greater than the one rectangle and less than the other, and hence the above relation continues to hold.

Rate of Change of Area. In the case considered on the opposite page, to find the instantaneous rate of change of the area A with respect to x, we must find $\frac{dA}{dx}$.

Letting the coordinates of P, any point on the curve, be x and y, and letting ΔA be the area increment corresponding to the increments Δx and Δy, we then have, as before,

$$y\,\Delta x < \Delta A < (y + \Delta y)\Delta x,$$

where $y\,\Delta x$ is the area of the inscribed rectangle and $(y + \Delta y)\,\Delta x$ is the area of the corresponding circumscribed rectangle.



Dividing the above inequality by Δx, we therefore have

$$y < \frac{\Delta A}{\Delta x} < y + \Delta y.$$

If we now let $\Delta x \to 0$, it is evident from the figure that

$$\Delta y \to 0 \text{ and } \Delta A \to 0.$$

Then, since $$y + \Delta y \to y,$$

we see that $\frac{\Delta A}{\Delta x}$, which always lies between y and $y + \Delta y$, must approach y as its limit, and hence

$$\lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = y,$$

or

$$\frac{dA}{dx} = y.$$

That is, the variable ordinate MP, or y, is a measure of the rate of change of the area A with respect to x.

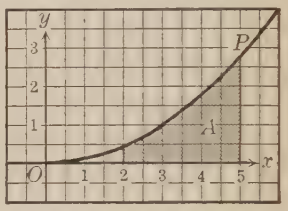
For example, if $\frac{dA}{dx} = 2$, we see that A is changing at the rate of 2 square units per linear unit of change in x.

If the equation of the given curve is $y = x^2$, we thus know that $\frac{dA}{dx} = x^2$, from which we can find the value of A by integration, as will be described on page 308.

Finding the Area under a Curve. The following examples illustrate the method outlined on pages 306 and 307:

1. Find an expression for the exact area under the curve $y = \frac{1}{9}x^2$ between a fixed ordinate at the point for which $x = 2$ and a variable ordinate at any point P on the curve.

The ordinate y of the variable point P measures the rate of change of the area A with respect to x (page 307); that is, we have



$$\frac{dA}{dx} = y.$$

Since $y = \frac{1}{9}x^2$,

then $\frac{dA}{dx} = \frac{1}{9}x^2$,

or $dA = \frac{1}{9}x^2\,dx$.

By integrating this expression for dA, we can now find the value of A: that is, we have

$$A = \int \tfrac{1}{9}x^2\,dx = \tfrac{1}{27}x^3 + c.$$

To determine c, the constant of integration, we see that when $x = 2$, we have $A = 0$, and hence we may write

$$\tfrac{1}{27} \cdot 2^3 + c = 0,$$

or $$c = -\tfrac{8}{27}.$$

The required expression for the area under the curve is therefore

$$A = \frac{x^3 - 8}{27}.$$

2. Find the area in Ex. 1 when P is the point on the curve for which $x = 5$.

Substituting 5 for x in the result of Ex. 1, we have

$$A = \frac{x^3 - 8}{27} = \frac{125 - 8}{27} = 4\tfrac{1}{3};$$

that is, there are $4\frac{1}{3}$ units of area under the curve. This result should be compared with the approximate result of Ex. 1 on page 302.

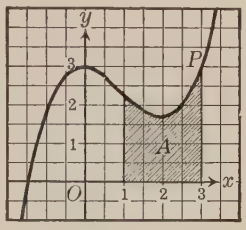
3. Find the area in Ex. 1 when P is the point $(2, \frac{4}{9})$.

We have $$A = \frac{2^3 - 8}{27} = 0;$$

that is, as is evident from the figure, the variable ordinate coincides with the fixed ordinate and there is no area under the curve.

4. Find the area under the curve $y = \frac{1}{3}x^3 - x^2 + 3$ between the ordinates at the points for which $x = 1$ and $x = 3$.

Drawing the graph of the equation and proceeding in the same way as in Ex. 1 on page 308, we have



$$\frac{dA}{dx} = y.$$

Hence $$\frac{dA}{dx} = \frac{1}{3}x^3 - x^2 + 3,$$

or $$dA = (\frac{1}{3}x^3 - x^2 + 3)dx.$$

Then $$A = \int (\frac{1}{3}x^3 - x^2 + 3)dx$$

$$= \frac{1}{12}x^4 - \frac{1}{3}x^3 + 3x + c.$$

To determine c, the constant of integration, we first observe that $A = 0$ when $x = 1$. We therefore have

$$0 = \frac{1}{12} - \frac{1}{3} + 3 + c;$$

whence $$c = -\frac{11}{4},$$

and $$A = \frac{1}{12}x^4 - \frac{1}{3}x^3 + 3x - \frac{11}{4}.$$

Since the variable ordinate of P is at the point for which $x = 3$, we then have

$$A = \frac{81}{12} - 9 + 9 - \frac{11}{4} = 4;$$

that is, the required area under the curve between the given ordinates contains 4 square units.

5. Solve Ex. 4 when the variable ordinate is at the point for which $x = 2$.

Substituting 2 for x in the result of Ex. 4, we have

$$A = \frac{16}{12} - \frac{8}{3} + 6 - \frac{11}{4} = \frac{23}{12} = 1\frac{11}{12};$$

that is, there are $1\frac{11}{12}$ square units of area under the curve when the variable ordinate moves to the point for which $x = 2$. As is evident from the figure, this area is less than half of that found in Ex. 4.

6. Solve Ex. 4 when the variable ordinate is at the point for which $x = 0$.

Substituting 0 for x in the result of Ex. 4, we have

$$A = \frac{1}{12} \cdot 0 - \frac{1}{3} \cdot 0 + 3 \cdot 0 - \frac{11}{4} = -\frac{11}{4} = -2\frac{3}{4};$$

that is, there are $2\frac{3}{4}$ square units of area in this case, the negative sign simply indicating that the area considered lies at the left of the fixed ordinate instead of at the right, as before.

Exercises. Areas

Find the area under the curve for each of the following cases:

1. $y = x^2$, between $x = 0$ and $x = 5$.

2. $y = -x^2 + 5x$, between $x = 2$ and $x = 5$.

3. $y = \frac{1}{27}x^3$, between $x = 1$ and $x = 3$.

4. $y = -x^3 + 4x^2$, between $x = 1$ and $x = 4$.

5. $y = -x^3 + 4x^2 + 2$, between $x = 2$ and $x = 4$.

6. $y = x^{\frac{1}{2}}$, between $x = 1$ and $x = 9$.

7. Find the area between the curve $y^2 = x^3$ and the double ordinate at the point for which $x = 4$.

One of these ordinates is positive and the other is negative. Upon drawing the graph the nature of the problem will be evident.

8. Draw the graph of the curve $y = x^3 - 16x$ and find the area under the part of the curve which lies in the second quadrant and also for the part of the curve which lies in the fourth quadrant.

9. Find the area of the segment of the parabola $y^2 = 9x$ cut off by the double ordinate at the point for which $x = 4$.

10. Find the area between the parabola $y^2 = 8x$ and its focal width.

11. Find the maximum and minimum ordinates of the curve $y = \frac{1}{3}x^3 - x^2 + 4$ and the area under the curve between these ordinates.

12. Find the area under the curve $y = x^3 + 4x^2 + 1$ between the maximum and minimum ordinates.

13. Draw the graph of the equation $y = -2x^2 + 9x - 4$, and find the area bounded by the curve and the x axis.

14. In Ex. 13 find the area under the curve between the ordinate at the point on the curve for which $x = 1$ and the maximum ordinate.

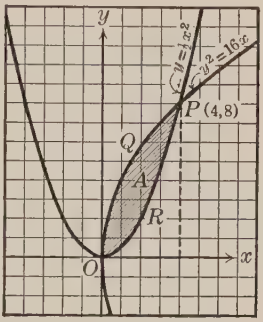
Area between Two Curves. The method explained on pages 308 and 309 can also be applied to finding the area between two curves.

For example, find the area between the two parabolas

$$y^2 = 16\,x$$

and

$$y = \tfrac{1}{2}\,x^2.$$



Drawing the graphs, as shown in the figure at the right, it is evident that the required area (the shaded portion marked A in the figure) can be found by subtracting the area under the curve ORP from that under the curve OQP.

Solving the two equations, we find that when $x = 0$, $y = 0$, and when $x = 4$, $y = 8$; that is, as shown by the figure above, the two curves intersect at the two points $O(0, 0)$ and $P(4, 8)$.

Letting A_1 represent the area under the parabola $y^2 = 16\,x$ (the curve OQP) and A_2 the area under the parabola $y = \frac{1}{2}\,x^2$ (the curve ORP) between the points for which $x = 0$ and $x = 4$, we then have

$$\frac{dA_1}{dx} = y = \sqrt{16\,x} = 4\,x^{\frac{1}{2}},$$

and

$$\frac{dA_2}{dx} = y = \tfrac{1}{2}\,x^2.$$

Letting A represent the area between the curves, we now have

$$A = A_1 - A_2,$$

and

$$\frac{dA}{dx} = \frac{dA_1}{dx} - \frac{dA_2}{dx} = 4\,x^{\frac{1}{2}} - \tfrac{1}{2}\,x^2;$$

whence

$$A = \int (4\,x^{\frac{1}{2}} - \tfrac{1}{2}\,x^2)dx = \tfrac{8}{3}\,x^{\frac{3}{2}} - \tfrac{1}{6}\,x^3 + c.$$

To determine c, the constant of integration, we first observe that $A = 0$ when $x = 0$. We therefore have

$$0 = \tfrac{8}{3} \cdot 0 - \tfrac{1}{6} \cdot 0 + c;$$

whence

$$c = 0.$$

Hence

$$A = \tfrac{8}{3}\,x^{\frac{3}{2}} - \tfrac{1}{6}\,x^3,$$

and when $x = 4$,

$$A = \tfrac{8}{3} \times 8 - \tfrac{1}{6} \times 64 = 21\tfrac{1}{3} - 10\tfrac{2}{3} = 10\tfrac{2}{3};$$

that is, the shaded portion between the curves contains $10\frac{2}{3}$ square units.

Exercises. Area between Curves

1. Find the area between the curve $y = \frac{1}{2} x^2$ and the straight line $2x - y = 0$.

2. Find the area between the curve $y^2 = x$ and the straight line $x = 2$.

3. Find the area between the curves $y^2 = 4x$ and $x^2 = 4y$.

4. Find the area of the segment of the parabola $y^2 = x$ cut off by the line $x = 4$.

5. Find the area between the two parabolas $y = \frac{1}{9} x^2$ and $y = \frac{1}{27} x^3$.

6. Find the area which is bounded by the curve $y^2 = x^3$ and the curve $y^2 = 4x$.

7. Find the area between the curve $y = -x^2 + 9x - 18$ and the straight line $y = 2$.

8. Find the area cut off from the curve $y = -x^2 + 6x - 5$ by the straight line $y - x + 1 = 0$.

9. Find the area between the curves $y = -x^2 + 9x - 18$ and $y = -x^2 + 6x - 8$ cut off by the x axis.

10. Draw the graphs of the equations $y = x^2 - 5x + 2$ and $y = -x^2 + 5x + 4$ and find the area between the two curves.

11. Find the area between the curves $y^2 = kx$ and $x^2 = ky$, where k is a constant.

12. Find the area between the curve $y^2 = x^3$ and the straight line $y = 2x$.

13. Find the area bounded by the curve $y = x^2 - 2x + 1$ and the coordinate axes.

14. In Ex. 13 find the area under the curve between the ordinates at the points at which the line $y = 9$ cuts the curve.

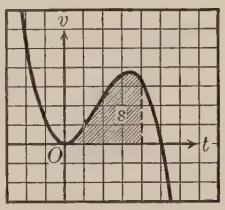
15. Find the area between the curve $y = 4x - x^2$ and the line drawn through the origin and the upper end of the maximum ordinate.

Distance Found by Integration. On page 303 we found that the distance traveled in t seconds by a body moving at a speed v is equal to the area under the curve which represents the relation between v and t.

On page 303 we found the approximate distance by using the Trapezoidal Formula in a case where the functional relation between v and t was not known.

For example, if for a certain particle which is starting from rest the relation between the speed v, in feet per second, and the time t, in seconds, is given by the equation

$$v = t^2 - 0.2\,t^3,$$



we see that the distance in feet, which is conventionally designated by either d or s, that the particle travels in 4 sec. is represented by the shaded area in the figure at the right.

As we found on page 307, the ordinates (v) of the curve measure the rate of change of the area (which is equal to the speed) with respect to t; that is,

$$\frac{ds}{dt} = v,$$

or

$$ds = v\,dt;$$

whence

$$s = \int v\,dt.$$

For the equation $v = t^2 - 0.2\,t^3$, given above, we therefore have

$$s = \int (t^2 - 0.2\,t^3)dt = \tfrac{1}{3}\,t^3 - 0.2 \times \tfrac{1}{4}\,t^4 + c.$$

Since the particle starts from rest, we know that $s = 0$ when $t = 0$, and hence $c = 0$; that is, the distance s traveled by the particle in t seconds is given by the equation

$$s = \tfrac{1}{3}\,t^3 - 0.05\,t^4.$$

Hence, when $t = 4$, we have

$$s = \tfrac{1}{3} \times 64 - 0.05 \times 256 = 8.533 \cdots;$$

that is, we thus find, to the nearest 0.01 ft., that the distance traveled by the particle in the 4 sec. is 8.53 ft.

Work. On page 304 we found that the work W done by a constant force F in moving a body a distance x is given by

$$W = Fx.$$

When the force varies with the distance, it can be shown, as on page 313, that the value of W is given by the formula

$$W = \int F\,dx.$$

For example, suppose that the relation between the force F, in pounds, required to stretch a spring and the distance x, in inches, that it stretches is given by the equation $F = 15\,x$. Find the work done in stretching the spring 4 in.

We have $$W = \int F\,dx = \int 15\,x\,dx = \tfrac{15}{2}x^2 + c.$$

Since $W = 0$ when $x = 0$, we find that $c = 0$, and hence

$$W = \tfrac{15}{2}x^2.$$

Then when $x = 4$, $$W = 120;$$

that is, the work done by the force is $\frac{1}{1}\frac{5}{2} \times 120$, or 10 ft.-lb.

Momentum. If a constant force F acts on a body for t seconds, the change in momentum M imparted by the force is given by

$$M = Ft.$$

As above, when the force varies with the time, the change in momentum is given by the formula

$$M = \int F\,dt.$$

For example, if the relation between the force F and the time t is given by $F = 20 - 0.3\,t^2$, what change in momentum is produced in the interval from $t = 1$ to $t = 3$?

We have $$M = \int F\,dt = \int (20 - 0.3\,t^2)dt = 20\,t - 0.1\,t^3 + c.$$

Regarding M as 0 when $t = 1$, we see that $c = -19.9$, and hence, when $t = 3$,

$$M = 60 - 2.7 - 19.9 = 37.4;$$

that is, the change in momentum due to the action of the force is 37.4.

Exercises. Problems from Physics

Find the distance traveled by an object in the specified time when the relation of the speed v to the time t is as follows:

1. $v = t^2 + 2t$; from $t = 2$ to $t = 6$.
2. $v = 2t^2 - 0.4t^3$; from $t = 0$ to $t = 2$.
3. $v = t^3 + t^2 + 3t$; from $t = 1$ to $t = 5$.
4. $v = \frac{5}{t^2}$; from $t = 2$ to $t = 4$.
5. $v = \sqrt{t} + \frac{2}{t^2}$; from $t = 1$ to $t = 4$.
6. $v = 3t^2 - 0.2t^3 + 1$; from $t = 0$ to $t = 3$.

Find the work done in moving an object through the specified distance when the force F is related to the distance x as follows:

7. $F = x^2 + 2x$; from $x = 1$ to $x = 5$.
8. $F = \frac{200}{x^2}$; from $x = 10$ to $x = 20$.
9. $F = \frac{400}{x^2}$; from $x = 2$ to $x = 6$.

Find the change in momentum caused in the specified time by a force F which is related to the time t as follows:

10. $F = 16 - t^2$; from $t = 0$ to $t = 3$.
11. $F = 3t^2 + 2t$; from $t = 2$ to $t = 5$.
12. $F = t^{\frac{1}{2}} + 5$; from $t = 1$ to $t = 9$.

13. If the relation between the velocity v and the time t of a body projected vertically upward is expressed by the equation $v = 1280 - 3t$, what is the height reached at the end of 10 sec.? What is the greatest height reached? When does the body return to the ground?

14. The formula which expresses the force F of attraction between two spheres x inches apart between centers is $F = \frac{25}{x^2}$. Find the work done in increasing x from 6 in. to 10 in.

Exercises. Review

Perform each of the following integrations:

1. $\int (5\,x^4 - 4\,x^3 + 6\,x^2 + 2\,x + 5)dx.$

2. $\int \left(\frac{a}{\sqrt{x}} + \frac{b}{x^2} + \frac{c}{x^3} - \sqrt{x} + 3\right)dx.$

3. $\int \left(\sqrt{2\,x} + \frac{3}{\sqrt{2\,x}} + 6(2\,x)^{\frac{3}{2}}\right)dx.$

4. $\int [x(2\,x^2 + 5\,x - 7)]\,dx.$

5. If $\frac{dy}{dx} = 6\,x^2 - 2\,x + 4$, and if $y = 3$ when $x = 1$, what is the equation that expresses the relation between x and y?

6. Find the area under the curve $y = \frac{1}{3}\,x^3 - x^2 + 3$ between its maximum and minimum ordinates.

7. The relation between the speed v and the time t for a certain particle is given by the equation $v = t^3 - 6\,t^2 + 36\,t$. Draw a graph showing how v varies with respect to t from $t = 0$ to $t = 4$. By the Trapezoidal Formula compute the approximate distance traveled during this time, and then find the precise distance by integration.

8. If the speed v of a particle starting from rest is given by the equation $v = 3\,t^2 + 20\,t + 16$, what is the distance of the particle from its starting point after t seconds? Find the distance traveled from $t = 2$ to $t = 8$.

9. Find the change in momentum caused by a force during the first 3 sec. when the relation of the force to the time is expressed by the equation $F = 30\,t - 0.2\,t^3$.

10. When the relation between the volume V and the pressure P of expanding steam is $PV^{0.9} = 8000$, how much work is done when the volume expands from 1 to 10?

The work done (W) is given by the expression $W = \int P\,dV$.

CHAPTER VII

NUMERICAL EQUATIONS

1. Polynomials

Special Definition of Polynomial. In this chapter we shall limit the meaning of the general term "polynomial" to include only *rational integral functions* of a variable; that is, those which can be formed by applying to the variable a finite number of times one or more of the operations of addition, subtraction, or multiplication.

That is, in the sense in which we shall now use the term, $x^2 + 2x - 5$ and $\frac{1}{2}x^3 + 1$ are polynomials, but $\frac{a}{x} + 1$ and $\sqrt{x} - a$ are not. Division by known numbers, however, is permitted.

The general form of a polynomial in x is

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where n is a positive integer and a_0, a_1, $a_2 \cdots$ are constants. Our work will chiefly concern cases in which the constants are rational.

We shall frequently represent polynomials by the function symbol $f(\ )$; that is, we shall write

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n.$$

We may also, if we wish, use the form

$$f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n,$$

which is known as the "p form" of the polynomial and is derived by dividing each term of the general form of the polynomial by a_0 and then letting

$$p_1 = \frac{a_1}{a_0},\ p_2 = \frac{a_2}{a_0},\ \cdots,\ p_n = \frac{a_n}{a_0}.$$

Remainder Theorem. When we substitute r for x in the general form of a polynomial,

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, \qquad (1)$$

we have $f(r) = a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n.$ (2)

Subtracting (2) from (1), we then have

$$f(x) - f(r) = a_0(x^n - r^n) + a_1(x^{n-1} - r^{n-1}) + \cdots,$$

in which $x - r$ is one factor of the second member and some function of x, let us call it $Q(x)$, is the other factor.

That is, $f(x) - f(r) = (x - r) \cdot Q(x)$;

whence $$\frac{f(x)}{x - r} = Q(x) + \frac{f(r)}{x - r}.$$

Hence, when $f(x)$ is divided by $x - r$, the quotient is $Q(x)$ and the remainder is $f(r)$; that is, we have the following law, which is known as the *Remainder Theorem*:

If $f(x)$ is divided by $x - r$, the remainder is $f(r)$.

For example, in dividing $2x^2 + 3x + 6$ by $x - 2$ by using both actual division and the Remainder Theorem, we have

Actual Division	Remainder Theorem
$2x^2 + 3x + 6 \mid x - 2$	Substituting 2 for x in
$2x^2 - 4x \mid 2x + 7$	$f(x)$, we have
$7x + 6$	$f(2) = 2 \cdot 2^2 + 3 \cdot 2 + 6$
$7x - 14$	$= 8 + 6 + 6$
20 (remainder)	$= 20$ (remainder)

We thus see that the remainder in each case is 20; that is, $x - 2$ is not a factor of $2x^2 + 3x + 6$. The Remainder Theorem can often be used to determine this fact mentally, and it gives the result more quickly than by actual division.

From the Remainder Theorem it follows that if $f(r) = 0$, then r is a root of the equation $f(x) = 0$.

For example, we see that 1 and -3 are the roots of the equation $x^2 + 2x - 3 = 0$, because $f(1) = 1 + 2 - 3 = 0$ and $f(-3) = 9 - 6 - 3 = 0$.

Evaluating $f(x)$. Instead of evaluating $f(x) = 2x^2 + 3x + 6$ for $x = 2$ by the method used on page 318, we might have proceeded as follows:

$$\begin{array}{r} 2x^2 + 3x + 6 \\ \underline{4x} \\ 7x + 6 \\ \underline{14} \\ 20 \end{array}$$

That is, when $x = 2$ we see that $2x^2 = 2 \cdot x \cdot x = 4x$, and we therefore add $4x$ to $3x + 6$. Then $7x = 7 \cdot 2 = 14$, and we add 14 to the 6, thus finding that $f(2) = 20$.

It is much easier, however, if we detach the coefficients of $f(x)$ and write simply

$$\begin{array}{r|l} 2 + 3 + 6 & \underline{2} \\ \underline{4 + 14} & \\ 2 + 7 + \mathbf{20} & \end{array}$$

in which the heavy type indicates that $f(2) = 20$.

That is, in this method we first bring down the coefficient 2; then $2 \times 2 = 4$, and we add 4 to the 3; then $2 \times 7 = 14$, and we add 14 to the 6. The last result, 20, is the value of $f(x)$.

As was shown on page 318, evaluating $f(x)$ when $x = r$ is the same as finding the remainder in dividing $f(x)$ by $x - r$. It will be observed that the line $2 + 7 + \mathbf{20}$ in the last example above gives the coefficients of the result of dividing $2x^2 + 3x + 6$ by $x - 2$, which, as found on page 318, is $2x + 7$, with the remainder 20.

We thus have not only an easy method of evaluating $f(x)$ for $x = r$, whatever the value of r or the number of terms in $f(x)$, but we also have an easy method of dividing $f(x)$ by $x - r$. When used for this latter purpose the method is known as *synthetic division.*

If we have to evaluate an expression like $f(x) = x^5 - 1$, we may write

$$f(x) = x^5 + 0x^4 + 0x^3 + 0x^2 + 0x - 1;$$

that is, the detached coefficients will be, in order, 1, 0, 0, 0, 0, and -1, and should always be written in using this method.

Exercises. Remainder Theorem

In each of the following cases evaluate the function for the specified value of the variable by actual substitution:

1. $x^2 + 3x + 1$; $x = 3$.
2. $3x^2 + 6x + 7$; $x = 5$.
3. $x^3 - 15x + 6$; $x = 3$.
4. $x^4 - 1$; $x = -1$.
5. $x^4 + 1$; $x = 1$.
6. $x^5 - 32$; $x = 2$.

By actual division and also by using the Remainder Theorem find the remainder in each of the following cases:

7. $\dfrac{x^3 - x^2 + x + 5}{x - 1}$.
8. $\dfrac{x^3 - x^2 + 2x - 5}{x + 1}$.
9. $\dfrac{x^3 + 2x^2 + 5}{x - 4}$.
10. $\dfrac{x^3 - 2x^2 + 3x + 2}{x + 2}$.

Using the method described on page 319, evaluate

11. $x^3 - 2x^2 + 3x + 5$ for $x = 2$.
12. $x^4 + 6x^3 - 7x^2 - 3x + 1$ for $x = 0.2$.
13. $x^5 - 3x^4 - 6x^3 + 5x^2 - 2x + 7$ for $x = 3$.

Using the method described on page 319, find the remainder in each of the following cases:

14. $\dfrac{x^3 + 3x^2 - 4x - 2}{x - 1}$.
15. $\dfrac{x^3 - 2x^2 + 5x - 1}{x + 1}$.
16. $\dfrac{x^3 - 2x + 2}{x - 3}$.
17. $\dfrac{x^4 - 2x^2 + 5}{x + 2}$.

In Ex. 15 we have $1 - 2 + 5 - 1 \mid -1$; that is, since the divisor must be of the form $x - r$, we have $x + 1 = x - (-1)$.

In each of the following cases show that one of the roots of the equation has the value specified:

18. $x^3 + 2x^2 + 4x + 3 = 0$; $x = -1$.
19. $x^3 - 7x^2 + 16x - 12 = 0$; $x = 3$.
20. $x^4 - 2x^3 - 21x^2 + 62x - 40$; $x = 4$.

Drawing the Graph of a Polynomial. As we have already learned, to draw the graph of a polynomial of the form

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

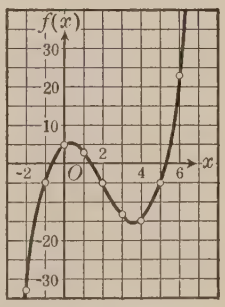
we may assign convenient values to x, compute the corresponding values of $f(x)$, and then locate the points which correspond to these pairs of values. For example, suppose that we wish to draw the graph of the polynomial

$$f(x) = x^3 - 6x^2 + 3x + 5.$$

In computing the values of $f(x)$ it is convenient to use the method described on page 319. The computations for the values of $f(x)$ corresponding to the values $-2, -1, \cdots, 6$ for x will then appear as follows, the heavy type indicating the values of x and $f(x)$ to be used in locating the points for the graph:

$1-6+3+5 \mid \underline{-2}$ $\quad -2+16-38$ $1-8+19-\mathbf{33}$	$1-6+3+5 \mid \underline{1}$ $\quad 1-5-2$ $1-5-2+\mathbf{3}$	$1-6+3+5 \mid \underline{4}$ $\quad 4-8-20$ $1-2-5-\mathbf{15}$
$1-6+3+5 \mid \underline{-1}$ $\quad -1+7-10$ $1-7+10-\mathbf{5}$	$1-6+3+5 \mid \underline{2}$ $\quad 2-8-10$ $1-4-5-\mathbf{5}$	$1-6+3+5 \mid \underline{5}$ $\quad 5-5-10$ $1-1-2-\mathbf{5}$
By inspection it is seen that when $x = 0$, $f(x) = \mathbf{5}$.	$1-6+3+5 \mid \underline{3}$ $\quad 3-9-18$ $1-3-6-\mathbf{13}$	$1-6+3+5 \mid \underline{6}$ $\quad 6\quad 0+18$ $1\quad 0+3+\mathbf{23}$

After locating the points which correspond to these pairs of values, we may sketch the curve shown at the right. On account of the large values of $f(x)$, it is convenient to take one space on the x axis equal to five spaces on the y axis, or the axis of $f(x)$.



From the graph of a polynomial we can make certain inferences that are of value in studying the polynomial. For example, in this case we can readily infer that $f(x)$ is maximum when x is between 0 and 0.5, and that $f(x)$ is minimum when x is between 3.5 and 4. On the following page we shall see that we can easily make other important inferences.

Studying the Graph of a Polynomial. As suggested on page 321, the graph of $f(x) = x^3 - 6x^2 + 3x + 5$ reveals that $f(x)$ is maximum when x is between 0 and 0.5 and minimum when x is between 3.5 and 4. For a closer approximation to these values of x, we may use the method which was described on page 283.

That is, taking $f'(x)$, the derivative of $f(x)$ with respect to x, we have

$$f'(x) = 3x^2 - 12x + 3.$$

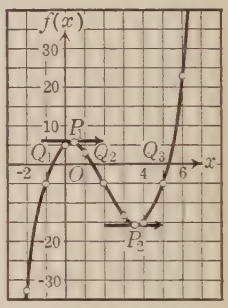
Hence $f(x)$ is maximum or minimum when

$$x^2 - 4x + 1 = 0.$$

Solving this equation, we then have

$$x = 2 \pm \sqrt{3} = 3.732 \text{ or } 0.268.$$

Now when $x = 0.268$, $f(x) = 5.392$, and when $x = 3.732$, $f(x) = -15.392$; that is, $f(x)$ is maximum at the point $P_1(0.268, 5.392)$ and minimum at the point $P_2(3.732, -15.392)$, as shown in the graph at the right.



We also see that the graph cuts the x axis at three points, which correspond to the values of x for which $f(x) = 0$. That is, we see that $f(x) = 0$ at Q_1, where x lies between -1 and 0; at Q_2, where x lies between 1 and 2; and at Q_3, where x lies between 5 and 6.

Methods of obtaining a closer approximation to these values of x will be discussed later.

Exercises. Graphs of Polynomials

Draw the graph of each of the following functions, find points for which $f(x)$ is maximum or minimum, and determine the integral values of x between which $f(x) = 0$:

1. $f(x) = x^3 - 2x^2 + 1$.
2. $f(x) = x^4 - 5x^2 + 4$.
3. $f(x) = 2x^4 - 8x + 6$.
4. $f(x) = x^3 - 3x + 2$.
5. $f(x) = 3x^3 - 12x + 5$.
6. $f(x) = 4x^3 - 12x + 7$.
7. $f(x) = x^3 + x^2 - 5x - 3$.
8. $f(x) = x^4 + 2x^3 - x^2 - 1$.

2. Theory of Equations

Roots of an Equation. Under the restriction applied to "polynomial" on page 317, it is evident that when a polynomial has the value zero we have a *rational integral equation.* That is, the general form of such an equation is

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0.$$

It can be shown that every rational integral equation has at least one root, but the proof is beyond the scope of our present work. We shall therefore simply assume that this statement is true.

The theorem which states that such an equation has at least one root is often called the Fundamental Theorem of algebra.

If the above equation has one root, for example r_1, we may divide both members by $x - r_1$, and the result will be an equation of degree $n - 1$. This equation will also have one root which we may designate as r_2. We may then divide both members of the equation by $x - r_2$, and the result will be an equation of degree $n - 2$. Continuing in the same way, it can then be shown that the general equation of the nth degree may be written as

$$(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0,$$

which has n roots. If there were any other root, we should have $n + 1$ factors, and the equation would not be of the nth degree.

We may therefore assume that

Every equation of the nth degree has n roots and no more.

These n roots may not all be different. For example, the equation $x^2 - 2x + 1 = 0$ has two roots, but each root is 1. This is apparent when we write the equation as $(x - 1)(x - 1) = 0$ or when we draw its graph and find that it is tangent to the x axis at $x = 1$.

The roots of the equation may also be fractional as well as integral, imaginary as well as real, irrational as well as rational, and one or more of the roots may be zero.

Removing Fractional Coefficients. In our work with equations we shall frequently find it desirable to reduce an equation to the p form (see page 317) with integral coefficients. For example, if we have the equation

$$2x^3 - x^2 + 3x + \tfrac{1}{3} = 0,$$

we may reduce it to the p form by dividing by 2. We then have

$$x^3 - \tfrac{1}{2}x^2 + \tfrac{3}{2}x + \tfrac{1}{6} = 0.$$

In order to remove the fractional coefficients, we may let $x = \frac{z}{m}$, and thus have

$$\frac{z^3}{m^3} - \frac{z^2}{2m^2} + \frac{3z}{2m} + \frac{1}{6} = 0,$$

or

$$z^3 - \tfrac{1}{2}mz^2 + \tfrac{3}{2}m^2z + \tfrac{1}{6}m^3 = 0.$$

We now select the least value for m that will make the coefficients integral; that is, if we let $m = 6$, we have

$$z^3 - 3z^2 + 54z + 36 = 0,$$

which is an equation in the p form with integral coefficients, as required.

The roots of this equation are not, of course, the same as those of the given equation in x, but since we know that $x = \frac{1}{6}z$, we can easily find the values of x from the values of z.

Exercises. Removing Fractional Coefficients

Reduce the following equations to the p form with integral coefficients:

1. $2x^3 + 3x^2 - 6x + 7 = 0.$
2. $3x^3 - x^2 + 6x - 9 = 0.$
3. $x^4 - \tfrac{1}{2}x^3 + \tfrac{2}{3}x^2 - x + 1 = 0.$
4. $x^4 - 2x^3 + \tfrac{3}{2}x^2 - 2x + \tfrac{1}{3} = 0.$
5. $3x^4 - 2x^3 + 6x^2 - x + 5 = 0.$
6. $2x^5 - 3x^4 + 2x^3 + 2x^2 - x - 5 = 0.$

Rational Roots. In examining for rational roots the p form of the general equation of the nth degree,

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0,$$

where $p_1, p_2, \cdots, p_n$ are integers, let us first consider the possibility of fractional roots. Thus, if $\frac{h}{k}$, a rational fraction reduced to lowest terms, is a root of the above equation, it is evident that the equation will be satisfied when $\frac{h}{k}$ is substituted for x; that is, we have

$$\frac{h^n}{k^n} + p_1\frac{h^{n-1}}{k^{n-1}} + p_2\frac{h^{n-2}}{k^{n-2}} + p_3\frac{h^{n-3}}{k^{n-3}} + \cdots + p_n = 0.$$

Multiplying by k^{n-1} and transposing, we have

$$\frac{h^n}{k} = -p_1h^{n-1} - p_2h^{n-2}k - p_3h^{n-3}k^2 - \cdots - p_nk^{n-1}.$$

Since the right side of this equation is integral, the left side must also be integral. This is impossible, however, because h and k have no common factor since they are the terms of a fraction in lowest terms; that is,

A rational fraction cannot be a root of an equation in the p form with integral coefficients.

Since we cannot have a rational fractional root, if we now let r be a rational integral root of the general equation, above, we have, as before,

$$r^n + p_1r^{n-1} + p_2r^{n-2} + \cdots + p_{n-1}r + p_n = 0.$$

Dividing by r and transposing, we then have

$$\frac{p_n}{r} = -r^{n-1} - p_1r^{n-2} - p_2r^{n-3} - \cdots - p_{n-1}.$$

Since the right side of this equation is integral, the left side must also be integral, and it therefore follows that p_n contains r as a factor; that is,

In an equation in the p form with integral coefficients, every rational root is a divisor of the constant term.

Finding Rational Roots. In finding the rational roots of an equation, we may apply the discussion on page 325, as shown in the following examples:

1. In the equation $x^3 - x^2 - 4x - 6 = 0$ find the rational roots.

Since the equation is in the p form with integral coefficients, we know from page 325 that there are no rational fractional roots and that the rational integral roots are factors of 6.

Testing the various factors of 6 (that is, ± 1, ± 2, ± 3, ± 6) by synthetic division, we find that $x - 3$ is a factor of the left member; that is, 3 is a rational root.

We may therefore write the given equation as

$$(x - 3)(x^2 + 2x + 2) = 0,$$

and hence

$$x^2 + 2x + 2 = 0.$$

The roots of this equation, however, are $-1 + \sqrt{-1}$ and $-1 - \sqrt{-1}$, and hence are not rational.

2. In the equation $2x^3 + 9x^2 - 16x + 5 = 0$ find all the rational roots.

First reducing the given equation to the p form by dividing both members by 2, we have

$$x^3 + \tfrac{9}{2}x^2 - 8x + \tfrac{5}{2} = 0.$$

To remove fractional coefficients, we may let $x = \frac{z}{m}$ and have

$$\frac{z^3}{m^3} + \frac{9z^2}{2m^2} - \frac{8z}{m} + \frac{5}{2} = 0,$$

from which, multiplying both sides by m^3, we have

$$z^3 + \tfrac{9}{2}mz^2 - 8m^2z + \tfrac{5}{2}m^3 = 0.$$

Then, letting $m = 2$ in order to make the coefficients of this last equation integral, we have

$$z^3 + 9z^2 - 32z + 20 = 0.$$

Testing the various factors of 20, we find that $z - 2$ is a factor of $f(z)$, and hence 2 is one root of the equation in z; that is,

$$(z - 2)(z^2 + 11z - 10) = 0.$$

Solving

$$z^2 + 11z - 10 = 0,$$

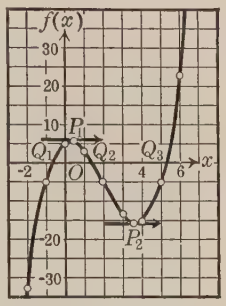
we then have

$$z = \tfrac{1}{2}(-11 \pm \sqrt{161});$$

that is, neither of the roots of this equation is rational. Since $x = \frac{z}{m} = \frac{1}{2}z$, we see that the only rational root of the given equation in x is 1.

Graphic Location of Real Roots. In our study of the graph of the polynomial $f(x) = x^3 - 6x^2 + 3x + 5$ on page 322, we found that the graph crossed the x axis at the three points for which $f(x) = 0$. That is, the abscissas of these points are the real roots of the equation

$$x^3 - 6x^2 + 3x + 5 = 0.$$



We here assume that the function is continuous for every value of x; that is, we assume that its graph is unbroken. The proof of this belongs to higher mathematics, and is beyond the scope of our present work.

We therefore locate one root as lying between $x = -1$ and $x = 0$, another as lying between $x = 1$ and $x = 2$, and the third as lying between $x = 5$ and $x = 6$.

That is, if in evaluating the polynomial $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, we find that $f(x)$ is positive when $x = q_1$ and negative when $x = q_2$, we conclude that a real root of the equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ lies between $x = q_1$ and $x = q_2$.

Exercises. Finding and Locating Roots

In Exs. 1–3 find four rational roots; in Exs. 4 and 5 find one rational root; and in Exs. 6 and 7 find two rational roots and locate the irrational ones:

1. $2x^4 + x^3 - 6x^2 + x + 2 = 0.$
2. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0.$
3. $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0.$
4. $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0.$
5. $x^3 + 63x - 316 = 0.$
6. $x^4 + 8x^3 + 9x^2 - 8x - 10 = 0.$
7. $x^4 - 2x^3 - 12x^2 + 10x + 3 = 0.$
8. One root of the equation $x^3 - 15x^2 - 33x + 847 = 0$ is 11. Find the other two roots.

Finding Irrational Roots. In examining an equation for irrational roots it is evident that the method of locating the roots described on page 327 is altogether too rough to be satisfactory. We shall first consider how a closer approximation may be obtained by the method of graphs, and shall then investigate other and more satisfactory methods of finding approximate roots.

For example, let us examine the equation $x^3 + x - 3 = 0$ for irrational roots.

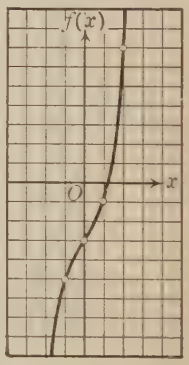
We may first locate roughly such irrational real roots as may exist by drawing a graph, as on page 327. In computing the values of $f(x)$ we may use synthetic division as follows, the heavy type indicating the pairs of values of x and $f(x)$ to be used in drawing the graph of the given equation:

$$\begin{array}{r|r} 1+0+1-3 & -2 \\ \quad -2+4-10 & \\ \hline 1-2+5-\mathbf{13} & \end{array} \qquad \begin{array}{r|r} 1+0+1-3 & 1 \\ \quad 1+1+2 & \\ \hline 1+1+2-\mathbf{1} & \end{array}$$

$$\begin{array}{r|r} 1+0+1-3 & -1 \\ \quad -1+1-2 & \\ \hline 1-1+2-\mathbf{5} & \end{array} \qquad \begin{array}{r|r} 1+0+1- & 3 \mid 2 \\ \quad 2+4+10 & \\ \hline 1+2+5+ & \mathbf{7} \end{array}$$

By inspection we see that when $x = 0$, $f(x) = -3$.

$$\begin{array}{r|r} 1+0+\ 1- & 3 \mid 3 \\ \quad 3+\ 9+30 & \\ \hline 1+3+10+ & \mathbf{27} \end{array}$$



Locating the points from $x = -1$ to $x = 2$, we may draw the graph here shown. If we should attempt to extend the graph for both positive and negative values of x, we should find that $f(x)$ continues to grow numerically larger. That is, although there are three roots, we have located only one, the one between $x = 1$ and $x = 2$, as is evident from both the graph and the table.

Taking $f'(x)$, the derivative of $f(x)$ with respect to x, we have

$$f'(x) = 3x^2 + 1.$$

Since $3x^2 + 1$ cannot be equal to zero for any real value of x, the curve has no maximum or minimum finite ordinates. That is, the curve cannot cut the x axis more than once, and hence the given equation has only one real root, the one between $x = 1$ and $x = 2$.

A Closer Approximation by Graphs. From the graph of the equation

$$x^2 + x - 3 = 0$$

on the opposite page we see that the one real root is nearer 1 than it is 2; that is, it lies between $x = 1$ and $x = 1.5$. In order to secure a closer approximation to this root we may increase the scale to which the graph is drawn. In this way we can magnify the portion of the graph between $x = 1$ and $x = 1.5$, the only part in which we are now interested.

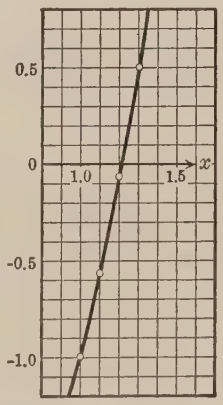
We may therefore give to x the values 1.1, 1.2, $\cdots$, 1.5 and compute the corresponding values of $f(x)$ as follows:

$$\begin{array}{r|l} 1+0 \quad +1 \quad -3 & 1.1 \\ \quad 1.1+1.21+2.431 & \\ \hline 1+1.1+2.21-0.569 & \end{array} \qquad \begin{array}{r|l} 1+0 \quad +1 \quad -3 & 1.2 \\ \quad 1.2+1.44+2.928 & \\ \hline 1+1.2+2.44-0.072 & \end{array}$$

$$\begin{array}{r|l} 1+0 \quad +1 \quad -3 & 1.3 \\ \quad 1.3+1.69+3.497 & \\ \hline 1+1.3+2.69+0.497 & \end{array}$$

Since $f(x)$ is negative when $x = 1.2$ and positive when $x = 1.3$, it is evident that the root must lie between $x = 1.2$ and $x = 1.3$, and hence we do not need to compute further values.

Letting each space represent 0.1, we may then draw the graph here shown, which is simply the portion of the graph on page 328 between $x = 1$ and $x = 1.3$ enlarged ten times.



The graph appears to cross the x axis about 0.1 space to the right of $x = 1.2$; that is, we thus estimate that 1.21 is a closer approximation for the root. Checking this value by synthetic division, we have

$$\begin{array}{r|l} 1+0 \quad +1 \quad -3 & 1.21 \\ \quad 1.21+1.4641+2.981561 & \\ \hline 1+1.21+2.4641-0.018439 & \end{array}$$

That is, since $f(x)$ is still negative the root is slightly larger than 1.21. Although we might continue this plan, and enlarge the portion of the graph between $x = 1.2$ and $x = 1.3$ to ten times its size in the above graph, it is evident that this method would soon become laborious. We shall therefore consider another and more precise method of finding the root.

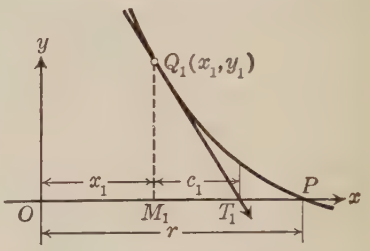
Newton's Tangent Method. The following method of finding an approximate value of an irrational root of an equation was suggested by Sir Isaac Newton (1642–1727).

For a real root of a rational integral equation of the form

$$y = f(x),$$

the curve will cross the x axis at some point P, as shown in the figure at the right. Then OP, or r, is the value of this root of the equation

$$f(x) = 0.$$



If x_1, the abscissa of a point $Q_1(x_1, y_1)$ on the curve, is taken as an approximate value of the root, the tangent at Q_1 will cut the x axis at some point T_1, as shown. In general, OT_1 will be a closer approximation to r than x_1 is.

From the point-slope formula on page 186, the equation of the tangent Q_1T_1 is

$$y - y_1 = m(x - x_1),$$

where m, the slope of the tangent, is the slope of the curve at Q_1, or the value of the derivative when $x = x_1$; that is,

$$m = f'(x_1).$$

Hence $$y - y_1 = f'(x_1) \cdot (x - x_1).$$

Then OT_1 is the value of x in this equation for which $y = 0$; that is, we have

$$x = x_1 - \frac{y_1}{f'(x_1)} = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Designating M_1T_1 in the figure as c_1, we then have

$$c_1 = OT_1 - x_1,$$

or

$$c_1 = -\frac{f(x_1)}{f'(x_1)},$$

which is a convenient formula for the correction c_1 to be added to x_1, which was the first approximation to r.

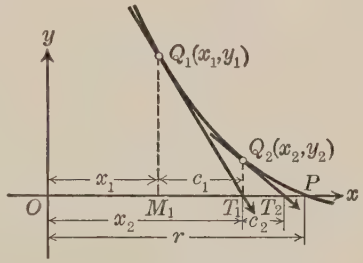
Second Correction. If we now take the point $Q_2(x_2, y_2)$ on the graph so that its abscissa x_2 is equal to $x_1 + c_1$, and draw the tangent Q_2T_2, as shown in the figure below, OT_2 will in general be a closer approximation to OP than OT_1 was.

Just as for the first correction, c_1, to be applied to the approximation x_1, we had

$$c_1 = -\frac{f(x_1)}{f'(x_1)},$$

so for the second correction, c_2, we now have

$$c_2 = -\frac{f(x_2)}{f'(x_2)},$$



and similarly for c_3, c_4, $\cdots$. If $f'(x) = 0$ or $f''(x) = 0$ in the vicinity of P, this method is likely to fail.

Newton's Method Applied. In applying this method to the equation

$$x^3 + x - 3 = 0,$$

considered on pages 328 and 329, we see that from the graph on page 329 we may take our first approximation (x_1) as 1.2.

Taking $f'(x)$, the derivative of the above equation, we have

$$f'(x) = 3x^2 + 1.$$

Then for the first correction, c_1, we have

$$c_1 = -\frac{f(x_1)}{f'(x_1)} = -\frac{f(1.2)}{f'(1.2)} = -\frac{-0.072}{5.32} = 0.013\cdots.$$

Hence $x_2 = x_1 + c_1 = 1.2 + 0.013 = 1.213.$

For the second correction, c_2, we then have

$$c_2 = -\frac{f(x_2)}{f'(x_2)} = -\frac{f(1.213)}{f'(1.213)} = -\frac{-0.002229403}{5.414107} = 0.0004\cdots.$$

Since c_2 does not affect the third decimal place, we see that the root of $x^3 + x - 3 = 0$ between 1 and 2 is 1.213, correct to the nearest 0.001.

We can find $f(1.213)$ most easily by synthetic division.

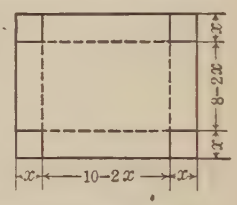
Exercises. Equations and Applications

By means of graphs locate the real roots of

1. $x^3 - 5x - 3 = 0$.
2. $8x^2 - 4x - 5 = 0$.
3. $x^3 - 6x^2 - 3x + 5 = 0$.
4. $x^3 + 2x^2 - 30x + 39 = 0$.

5. By Newton's method find to the nearest 0.001 one of the roots in each of Exs. 1–4.

6. An open box is to be made from a rectangular sheet of tin by cutting equal squares from the corner, as here shown, and folding up the flaps. If the piece of tin measures 8 in. by 10 in., how large a square should be cut from each corner so as to have the box contain 49 cu. in.?



7. The formula $V = \frac{1}{3}\pi(2r^3 + 3r^2x - x^3)$ gives the volume of a spherical segment with the altitude $x + r$ cut from a sphere of radius r. Find the altitude of a segment cut from a sphere with a radius of 5 in. in such a way that the volume of the segment is 0.8 of the volume of the sphere.

8. As in Ex. 7, find the altitude of a segment cut from a sphere with a radius of 7 in. in such a way that the volume of the segment is $\frac{3}{4}$ of the volume of the sphere.

9. The formula $\left(P + \frac{k_1}{V^2}\right)(V - k_2) = 1$ is frequently used in the study of gases. In this formula P represents the pressure and V the volume, and k_1 and k_2 are constants which vary for different gases. For a gas for which $k_1 = 0.009$, $k_2 = 0.003$, and $P = 1$, the above formula reduces to the cubic equation $V^3 - 1.003V^2 + 0.009V - 0.000027 = 0$. Letting $V = 0.01x$, we then have the equation $x^3 - 100.3x^2 + 90x - 27 = 0$. Solve this last equation for one value of x, and thus find one value of V.

CHAPTER VIII

PRACTICAL MENSURATION

1. Plane Figures

Nature of the Work. In a course like the one outlined in this book it is not feasible to give a satisfactory treatment of demonstrative solid geometry. One value of studying this subject is the training in logic thus received, but it is now generally recognized that for a general appreciation of the meaning of mathematics as a science an adequate course in demonstrative plane geometry, such as is presented in most high schools in this country, gives sufficient consideration to this phase of the subject.

A second and equally important value in studying solid geometry is the training that it gives in spatial perception. It is possible to cover this phase by a study of the mensuration of solid figures, although in the time that will generally be available it will not, of course, be possible to secure as complete results as can be obtained from the traditional course in solid geometry.

This chapter, therefore, presents a brief study of the mensuration of geometric figures. The first part, by way of introduction, deals with plane figures, and the second and larger part relates to solid figures, including a brief discussion of methods of representing figures in projection.

The preceding chapters, which cover the essential parts of algebra, trigonometry, analytic geometry, and the calculus, may properly be regarded as furnishing the body of the course. The present topic may therefore be regarded as optional, to be taken if time permits or the special needs of the class demand it.

Formulas. In practical mensuration extensive use is made of formulas. The following list may be referred to, as needed, in solving the exercises.

In the formulas the scheme of notation used is as follows:

A = area	d = diameter	$\pi = 3\frac{1}{7}$, or 3.1416
a, b, c = sides of $\triangle ABC$	h = altitude	r = radius
b, b' = bases	k = length of chord	s = side, semiperimeter
C = circumference	p = apothem	θ = central angle

When s is used for the semiperimeter, we have $s = \frac{1}{2}(a + b + c)$.

The formulas printed in heavy type are the most important, and, if not already familiar, they should be memorized.

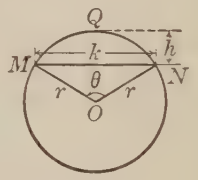
Circle, area, $\boldsymbol{A = \pi r^2}$ or $\boldsymbol{A = \frac{1}{4}\pi d^2}$

chord MN, $k = 2r \sin \frac{1}{2}\theta$

circumference, $\boldsymbol{C = 2\pi r}$ or $\boldsymbol{C = \pi d}$

sector $MONQ$, $A = \frac{1}{360}\pi r^2\theta$

segment MNQ, $A = \frac{2}{3}hk + \dfrac{h^3}{2k}$



The formula for the segment of a circle gives only approximate results, but they are sufficiently accurate for most purposes.

Unless it is otherwise specified, the value $3\frac{1}{7}$ may be used for π.

Hexagon, $A = 2.5980\, s^2$ $r = s$ $p = 0.8660\, s$

Octagon, $A = 4.8284\, s^2$ $r = 1.3065\, s$ $p = 1.2071\, s$

Parallelogram, $\boldsymbol{A = bh}$

Pentagon, $A = 1.7205\, s^2$ $r = 0.8506\, s$ $p = 0.6882\, s$

Trapezoid, $\boldsymbol{A = \frac{1}{2}h(b + b')}$

Triangle, area, $\boldsymbol{A = \frac{1}{2}bh}$ or $\boldsymbol{A = \sqrt{s(s-a)(s-b)(s-c)}}$

equilateral, $A = 0.4330\, b^2$ $h = 0.8660\, b$

right ($\angle C = 90°$), $\boldsymbol{c = \sqrt{a^2 + b^2}}$

The formulas for the hexagon, octagon, and pentagon refer to regular figures (that is, figures which are both equilateral and equiangular) with the apothem p, the radius r, and the side s.

Exercises. Mensuration of Plane Figures

1. Construct a triangle in which the sides are respectively 2.3 in., 3.2 in., and 3.8 in. long. Measure any altitude and find the area of the triangle.

In this chapter the dimensions are all given decimally, in accordance with the modern tendency. If rulers graduated to tenths of an inch are not at hand, a temporary substitute can be devised from graph paper ruled with ten squares to the inch.

2. Check the area found in Ex. 1 by means of the formula for A in terms of a, b, c, and s.

3. Draw a circle with a radius of 4.8 in. and take a point 8 in. from the center. Find the length of a tangent from this point to the circle, and draw both tangents.

The points of tangency are easily located by drawing an arc with the length of the tangent as radius.

4. Construct an equilateral triangle which will have an area of 3 sq. in. Check the work by measuring the base and altitude and computing the area from these measurements.

5. Find the area of a parallelogram in which the adjacent sides are respectively 2 in. and 3 in. long, and the included angle is 38° 20′. Check the result by drawing the figure, measuring the altitude, and computing the area.

The problem is first solved, of course, by trigonometric methods.

6. If the diagonal of a rectangle is 3.6 in. long and makes an angle of 32° with the base, what is the area of the rectangle? Check the result by drawing the figure, measuring the sides, and computing the area.

7. Find the area of a regular hexagon inscribed in a circle that is 2.8 in. in diameter.

8. In a rhombus with diagonals of 2.5 in. and 1.7 in., find the length of each side, the size of each angle, and the area.

A *rhombus* is a parallelogram in which all the sides are equal.

9. Find the length of the side of a regular octagon which is inscribed in a circle 3.5 in. in diameter.

10. A draftsman wishes to draw a regular pentagon that will be 1.8 in. on a side. What radius should he use in drawing the circle in which to inscribe the pentagon? What is the area of the pentagon?

11. The area of a regular octagon 1.6 in. on a side is what per cent greater than that of a regular hexagon with the same length of side? Is the same true if a regular octagon and a regular hexagon are each inscribed in a circle 1.6 in. in diameter? If not, what is the per cent in that case?

12. In a hexagonal nut which is 0.625 in. on a side, what is the "distance across the corners"? What is the "distance across the flats"?

The quoted terms, the meanings of which are obvious, are often used by machinists.

13. What is the cross-section area of the largest square timber that can be sawed from a round log 2 ft. 3 in. in diameter? from one with half this diameter?

14. Within a square which is 2 in. on a side inscribe the largest possible equilateral triangle and find its area.

The vertices of the triangle must lie on the sides of the square.

15. In measuring a city lot in the form of a trapezoid a surveyor finds that the parallel sides are respectively 128.3 ft. and 86.2 ft. long, while one of the non-parallel sides, which makes an angle of 78° 20′ with the longer parallel side, is 72.4 ft. long. Find the area of the lot, the length of the fourth side, and the size of each angle.

16. A sign painter who is laying out a clock face wishes to show the time as 18 min. past 8 o'clock. Draw a circle with a radius of 2 in. and show accurately the position in which the hands should be drawn.

17. A water main is to be laid to two branch pipes which have diameters of 10 in. and 16 in. respectively, and the diameter of the main must be such that the area of its cross section is equal to the combined cross-section areas of the two branch pipes. Find the diameter of the main to the nearest 0.25 in.

18. When a sector of 28.2° (that is, one in which the central angle contains 28.2°) is cut from a circle 6.3 in. in diameter, by how many square inches is the area of the circle reduced?

19. In a circle 12.5 in. in diameter find the length of the chord joining the ends of the radii which form a central angle of 32.8°.

20. When the water in a horizontal pipe 20 in. in diameter is 7.5 in. deep, what is the approximate cross-section area of the "wetted segment"?

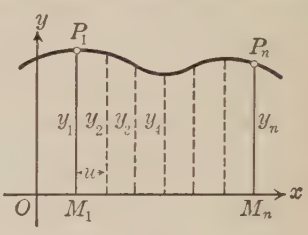
21. If the circumference of one circle is four times that of another, an equilateral triangle constructed on the diameter of the first as a side has how many times the area of an equilateral triangle constructed on the diameter of the second as a side?

22. Solve Ex. 21 when the area of the first circle is four times that of the second.

23. The formula $s' = \dfrac{2rs}{\sqrt{4r^2 - s^2}}$ gives the length s' of the side of a regular polygon circumscribed about a circle of radius r in which a regular polygon of the same number of sides, each of which has the length s, is inscribed. Find the length of the side of a regular octagon circumscribed about a circle 3.8 in. in diameter.

24. Two rectangular city lots have the same depth, the frontage of the first is twice that of the second, and their combined frontage is equal to their common depth. Find the ratio of their areas and the ratio of their perimeters.

Simpson's Rule for Areas. On page 302 we learned one formula for finding the approximate area under a curve. The formula given below furnishes another practical method of computing such an area when the equation of the curve is not known.



By drawing and measuring an *odd* number of ordinates (y_1, y_2, y_3, $\cdots$, y_n) having a common distance u between them, as here shown, we may compute the approximate area under the curve between the ordinates y_1 and y_n by means of the formula

$$A = \tfrac{1}{3}u\,[(y_1 + y_n) + 2(y_3 + y_5 + y_7 + \cdots) + 4(y_2 + y_4 + y_6 + \cdots)].$$

This formula expresses what is known as *Simpson's Rule*, so named from its inventor. Since there must be an odd number of ordinates, n must be an odd integer. The odd-numbered ordinates are therefore y_1, y_3, y_5, $\cdots$, y_n, while the even-numbered ordinates are y_2, y_4, y_6, $\cdots$.

Exercises. Simpson's Rule

1. If in a curve like the one above the ordinates y_1 to y_7 are respectively 2.00 in., 1.92 in., 1.76 in., 1.62 in., 1.78 in., 1.90 in., and 1.85 in., and if $u = 0.4$ in., what is the approximate area between y_1 and y_7?

2. Solve Ex. 1 by the Trapezoidal Formula given on page 302 and compare the two results.

3. Suppose that the above figure represents a field bounded on one side by a stream. If at intervals of 20 ft. the ordinates are found to be 98.2 ft., 106.7 ft., 128.4 ft., 126.2 ft., 124.6 ft., 122.7 ft., 108.4 ft., 108.4 ft., 106.3 ft., 108.5 ft., and 110.6 ft., what is the area of the field?

4. On squared paper draw an undulating curve, and by estimating the ordinates find the approximate area under the curve between the ordinates at $x = 4$ and $x = 18$.

2. Solid Figures

Formulas. The following list of formulas relating to solid figures supplements the list given on page 334, and should also be referred to in solving the exercises.

In addition to the notation on page 334, we shall use the following:

B = area of base $\quad S$ = area of curve surface

l = slant height $\quad V$ = volume

Cone, $\quad S = \pi rl \quad V = \frac{1}{3}\pi r^2 h$

Cylinder, $\quad S = 2\pi rh \quad V = \pi r^2 h$

Only right circular cones and cylinders, or cones and cylinders of revolution as they are often called, will be considered. As on page 334, the important formulas are printed in heavy type.

Frustum of a cone,

$$S = \pi l(r + r') \qquad V = \tfrac{1}{3}\pi h(r^2 + r'^2 + rr')$$

Here r' is used to represent the radius of the upper base.

Frustum of a pyramid,

$$V = \tfrac{1}{3}h(B + B' + \sqrt{BB'})$$

Prism, $\quad V = Bh$

Pyramid, $\quad V = \frac{1}{3}Bh$

It is unnecessary to give formulas for the lateral area of a prism or the area of the total surface of a cone or a cylinder, as these areas can be computed by taking the sum of areas found from formulas already given.

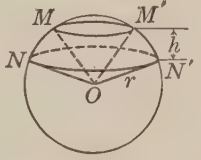
Sphere, $\quad S = 4\pi r^2 \quad V = \frac{4}{3}\pi r^3$

sector O-$MNN'M'$, $\quad V = \frac{1}{3}rS$

segment $MNN'M'$,

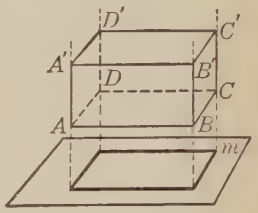
$$V = \tfrac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2)$$

zone, $\quad S = 2\pi rh$



In the formula for the sector the letter S represents the area of the zone $MNN'M'$ which forms the base, and in the formula for the segment the letters r_1 and r_2 represent the respective radii of the two bases.

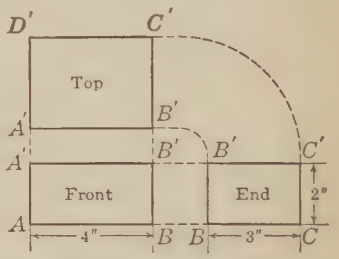
Orthogonal Projection. Solids are represented as if projected upon a plane. We may imagine these projections as the shadows of wire models of the solids cast by the sun's rays, which for practical purposes we may regard as parallel. If, as here shown, the rays are perpendicular to the plane of projection, we have what is known as *orthogonal projection.*



The word "orthogonal" means right-angled. In the above figure the plane of projection m is parallel to the base $ABCD$ of the model of the rectangular solid, and hence the projection is simply a rectangle with the same dimensions as the base of the model.

If we take the orthogonal projections of a rectangular solid upon planes parallel to the top, the front, and the end respectively, we shall have, as shown in the figure below, a top, a front, and an end view of the solid which show these faces in their correct proportions.

That is, if the solid is 4 in. long, 3 in. wide, and 2 in. high, in a full-size drawing the faces shown in this figure will have these dimensions, and all angles will be right angles.

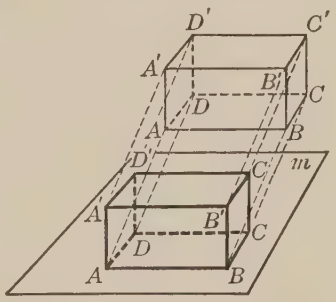


In such a case the measurement of an angle or of a line in the projection will yield a true measure of the corresponding angle or line in the solid.

In a perspective drawing, like the one at the top of the page, the parallel edges like BC and $B'C'$, which recede from the observer, are made to converge, and the face angles, like $\angle D'C'B'$, are made oblique instead of right angles. It is evident that measurements on such a drawing will not give the true dimensions of the solid.

A practical illustration of perspective is the apparent convergence of railroad tracks as they recede toward the horizon.

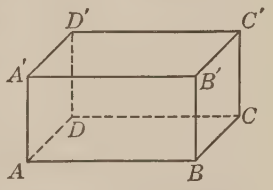
Oblique Projection. If the rays which cast the shadow of the model are oblique to the plane of projection, as shown in this illustration, the figure formed on the plane is said to be *in oblique projection.*



In this case the plane of projection m is parallel to the base $ABCD$ of the model. It is not necessary that the plane of projection be taken parallel to any face of the solid, but if this is not done, the resulting figure will be very much distorted.

If we wish to represent a rectangular solid that is 4 in. long, 3 in. wide, and 2 in. high in oblique projection in such a way that we may measure lines and angles in the drawing, a convenient method is as follows:

First draw the face $DCC'D'$ to represent a rectangle 4 in. long and 2 in. high, and then from the vertex C draw a line making an angle of 45° with DC. On this line lay off CB to represent $1\frac{1}{2}$ in. (that is, half its true length in the solid), and from the other vertices lay off $C'B'$, $D'A'$, and DA equal and parallel to CB. Then draw the lines AB, BB', $B'A'$, and $A'A$, thus completing the figure.



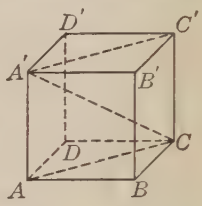
It should be observed that every right angle in this type of projection is represented by an angle of either 45°, 90°, or 135°, and that receding lines like BC and $B'C'$ are half their true length in the solid, thus making it possible to measure the drawing.

This type of oblique projection, which is sometimes called *cabinet projection*, will be used in the exercises to follow. The student will be expected to be able to reproduce the figures there given and to draw others of the same general nature.

It is sometimes convenient to take $\angle DCB$ as 30° or 60°, and to take BC as three fourths its true length in the solid, but it is not often necessary to do this.

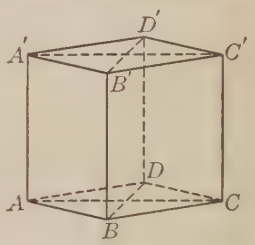
Exercises. Mensuration of Solid Figures

1. This figure shows a cube in oblique projection, as described on page 341. Make a full-size drawing showing the orthogonal projection of a cube 1.8 in. on an edge upon a plane parallel to the base. Find by measurement the length of the diagonal AC of the base and check the result by computing the length from the given dimensions.



2. In Ex. 1 draw in correct proportions the section of the cube formed by passing a plane through the edges AA' and CC'. Find by measurement the length of the diagonal $A'C$ and check the result by computation.

3. This figure shows the oblique projection of a right prism in which the base is a rhombus. Taking each side of the base as 12 in., $\angle BAD$ as 56°, and the height as 16 in., draw to a convenient scale the orthogonal projection of the prism upon a plane parallel to the base. Find by measurement the lengths of the diagonals AC and BD and check the result by computation.



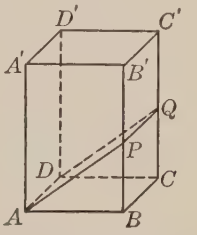
4. In Ex. 3 compute the volume of the prism and the area of its total surface.

5. In Ex. 3 draw to scale the orthogonal projection of the prism upon a plane parallel to a plane through BD and $B'D'$. Find the length of the shortest diagonal of the prism.

A diagonal of a prism is a line that joins any two vertices which are not in the same face.

6. In Ex. 3 draw to scale the orthogonal projection of the prism upon a plane parallel to a plane through AC and $A'C'$. Find the length of the longest diagonal of the prism.

7. This figure shows a right prism with a square base in oblique projection. Taking the height of the prism as 8 in. and the side of the base as 5.2 in., find the dimensions of the section $APQD$ formed by passing a plane through AD at an angle of 32° with the base of the prism.

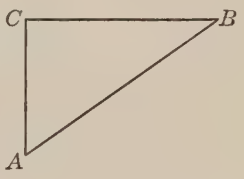


Since the faces $ABB'A'$ and $DCC'D'$ are shown in their correct proportions, $\measuredangle BAP$ and CDQ in the figure actually contain 32°.

8. In Ex. 7 show that the area of the section $APQD$ is equal to the area of the base $ABCD$ divided by $\cos BAP$.

9. In Ex. 7 compute the volume of the triangular prism ABP-DCQ and of the truncated prism $A'B'C'D'$-$APQD$.

10. This figure shows the orthogonal projection of a triangular right prism upon a plane parallel to the base. Taking the length of the edge BC as 3 in., $\angle B$ as 35°, and $\angle C$ as 90°, and taking the height of the prism as 4.5 in., draw the prism in oblique projection, as described on page 341. Show in your figure a plane passing through the edge AC at an angle of 25° with the base of the prism, and compute the area of the total surface of the truncated prism thus formed.

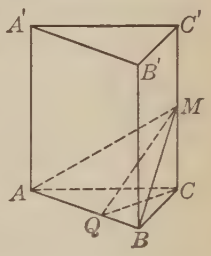


11. In Ex. 10 compute the volume of the triangular pyramid cut off by the plane through AC, and from this find the volume of the truncated prism.

12. In Ex. 10 find the angle with the base at which a plane should pass through AC in order to make the volume of the triangular pyramid cut off equal to half the volume of the truncated prism that is left.

13. In Ex. 12 compute the area of the total surface of the triangular pyramid cut off by the plane.

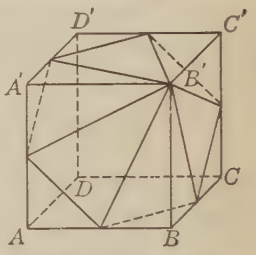
14. This figure shows in oblique projection a right triangular prism in which the faces AC' and BC' meet at right angles. Taking AC as 4 in., BC as 3 in., and CC' as 4.5 in., find by the method of Ex. 8 the area of the section ABM formed by passing a plane through AB and the midpoint M of the edge CC'.



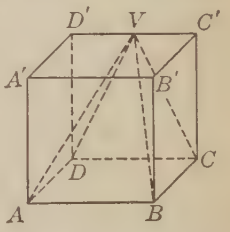
We need to find first the angle at which the plane meets the base. This angle is the $\angle CQM$, which is formed by drawing lines from C and M perpendicular to AB. Drawing the base ABC in correct proportions will show how to find the length of CQ.

15. In Ex. 14 find the angle with the base at which a plane should be passed through AB and some point on the edge CC' in order to cut off one fourth the volume of the prism.

16. This figure shows how a regular hexagonal pyramid may be inscribed in a cube by joining the midpoints of certain edges to each other and to one of the vertices. Find the volume and the total-surface area of such a pyramid inscribed in a cube 8.2 in. on an edge.

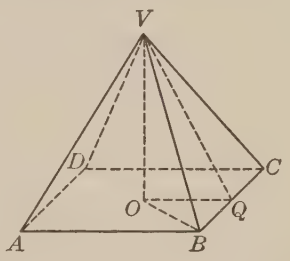


17. This figure shows in oblique projection a cube in which is inscribed a pyramid with the midpoint V of the edge $C'D'$ as the vertex. Compute the volume and the total-surface area of such a pyramid inscribed in a cube that is 3.6 in. on an edge.



18. In Ex. 17, how does the pyramid formed by joining B, C, C', and B' to the vertex A compare with the pyramid V-$ABCD$ in respect to volume? Which of the two pyramids has the larger total-surface area, and how much larger is this area than the other?

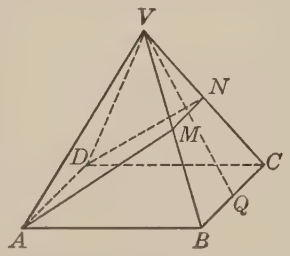
19. This figure shows in oblique projection a regular pyramid with a square base. Compute the volume of such a pyramid in which the base is 3.8 in. on a side and the height is 3.5 in.



20. In Ex. 19 compute the length of a lateral edge (as BV) and the angle which it makes with the base.

21. In Ex. 19 compute the area of the total surface.

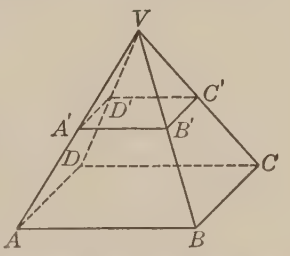
22. This figure shows the pyramid of Ex. 19 with a plane through AD and the midpoints M and N of the lateral edges VB and VC. Find the angle which the plane makes with the base and the area of the section $AMND$.



Draw in correct proportions the section of the pyramid formed by passing a plane through VQ parallel to AB. This section will show both the required angle and the altitude of the trapezoid $AMND$.

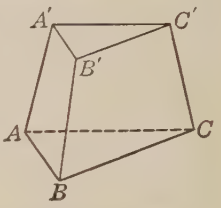
23. In Ex. 22 show why the method suggested in Ex. 8 cannot be used in finding the area of the section $AMND$.

24. This figure shows the pyramid of Ex. 19 with a plane passed through the respective midpoints A', B', C', and D' of the lateral edges VA, VB, VC, and VD. Find the ratio of the areas of the squares $A'B'C'D'$ and $ABCD$.



25. In Ex. 24 find the volume of the pyramid V-$A'B'C'D'$, and from it find the volume of the frustum of the large pyramid formed by cutting off the small pyramid. Check the result by finding the volume of this frustum by means of the formula given on page 339.

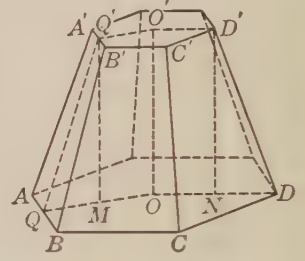
26. This figure shows in oblique projection the frustum of a regular triangular pyramid. Compute the volume of such a frustum in which the lower base is 5 in. on a side, the upper base is 3 in. on a side, and the height of the frustum is 3 in.



In this case use the formula on page 339.

27. In solid geometry it is proved that the volumes of two similar solids are proportional to the cubes of corresponding lines of the solids. What is the length of edge of the upper and lower bases of a frustum of a regular triangular pyramid that has twice the volume of the one in Ex. 26?

28. This figure shows in oblique projection the frustum of a regular hexagonal pyramid in which the lateral edges make an angle of 70° with the lower base. Find the altitude of such a frustum in which the edge of the lower base is 12 in. and that of the upper base is 6 in.



In the section $O'ODD'$, $\angle D'DO = 70°$, and $ON = O'D' = C'D'$.

29. In Ex. 28 compute the volume by the formula given on page 339, and check the result by finding the volume of the entire pyramid and subtracting the volume of the small pyramid which was cut off to form the frustum.

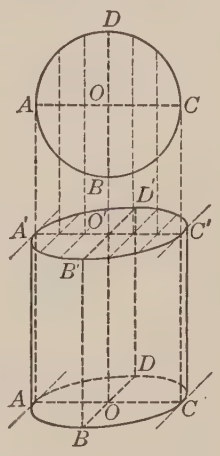
30. In Ex. 28 compute the slant height QQ' and the angles which the lateral faces make with both bases.

In the section $O'Q'QO$ we have $OM = O'Q'$, which is the apothem of the upper base.

31. In Ex. 28 compute the area of the total surface.

32. In Ex. 28 find the distance above the lower base at which a plane parallel to the bases should be passed in order to divide the frustum into two parts equal in volume.

33. This figure shows at the top the orthogonal projection of a right circular cylinder upon a plane parallel to the bases. Below is shown the cylinder in oblique projection. Find the area of the total surface and the volume of a cylinder that is 3.5 in. in diameter and 4 in. high.

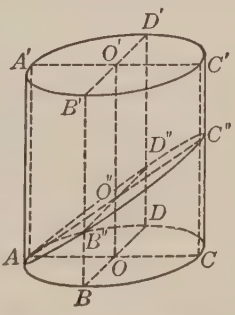


The dotted lines show how the ellipses which form the oblique projections of the circular bases are obtained. In the upper ellipse, $B'D'$ is half of BD in the orthogonal projection, and similarly each chord that crosses the axis $A'C'$ is half the corresponding chord in the circle. It will be sufficient for the student to locate the positions of the points A', B', C', and D' and then to draw the ellipse freehand.

34. A regular hexagonal prism is inscribed in a cylinder 4.2 in. in diameter and 5 in. high. Draw the figure in oblique projection and find the volume of the prism.

35. In Ex. 34 find, by the method of Ex. 8, the area of a section of the prism made by a plane cutting the axis of the cylinder and forming an angle of 30° with the base.

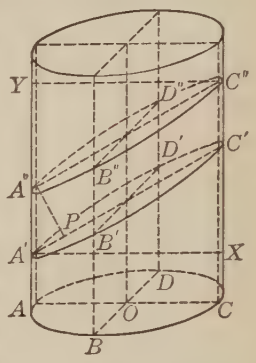
36. In Ex. 35 the inclined plane will cut the cylindric surface in an ellipse, as shown in this figure. Using the method of Ex. 8, find the area of this section when a cylinder 2.8 in. in diameter is cut by a plane inclined at an angle of 35° to the base.



37. The formula for the area A of an ellipse is $A = \pi ab$, where a and b are the major and minor semiaxes respectively. Check the result in Ex. 36 by means of this formula.

38. Solve Ex. 36 when the angle made by the plane is 45°.

39. In this figure, which shows a vertical cylinder cut by two parallel inclined planes, suppose that the diameter of the cylinder is 8.4 in., that the planes are inclined at an angle of 30° to the horizontal, and that $A'A'' = 4$ in. Find the volume of the segment between the planes.



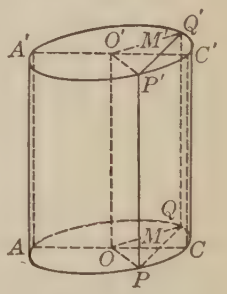
The altitude of the segment is the perpendicular $A''P$ between the planes.

40. In the figure of Ex. 39, the ellipse $A'C'$ bisects the cylindric surface in which the height is $C'X$, and the ellipse $A''C''$ bisects the cylindric surface with the height $A''Y$. Find the area of the curve surface of the segment between the two inclined planes.

41. A cylindric hollow column 6 in. in outside diameter and 12 ft. long is made of cast iron weighing 450 lb./cu. ft. If the metal is 1 in. thick, what is the weight of the column?

42. In Ex. 41 what is the area of the metal in a section made by a plane cutting the axis at an angle of 45°?

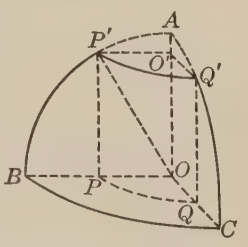
43. This figure shows in oblique projection a cylinder cut by a plane parallel to the axis. Find the volume of the smaller cylindric segment cut off by a plane parallel to the axis at a distance of 2 in. in a cylinder 6.3 in. in diameter and 8 in. high. What is the volume of the other segment?



The length of the chord PQ and the area of the circular segment PCQ may be found from the formulas given on page 334.

44. A horizontal cylindric tank 5.5 ft. in diameter and 18 ft. long is filled with gasoline to a depth of 20 in. At 7.5 gal./cu. ft., what is the amount of gasoline in the tank?

45. This figure shows one eighth of a sphere through which a cylindric hole has been bored, the axis of the hole passing through the center of the sphere, and the diameter of the hole being half that of the sphere. Find the length of such a hole bored through a sphere 5.6 in. in diameter.



That is, the required length is $2\ PP'$. Since $OP = \frac{1}{2}\ OP'$, what is the size of $\angle P'OP$? How can we then find PP'?

It is frequently more convenient in oblique projection to show only an eighth of the complete figure.

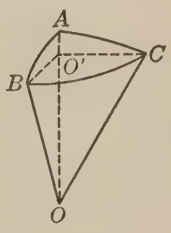
46. In the figure of Ex. 45, $A\text{-}P'Q'O'$ is one fourth of one of the two equal spherical segments which are cut off in boring the hole through the sphere. Find the volume of one of these segments.

In the formula $V = \frac{1}{6}\pi h(3\ r_1^2 + 3\ r_2^2 + h^2)$ we have $r_2 = 0$ in this case.

47. Another method of finding the volume of a spherical segment of one base, like those mentioned in Ex. 46, is to consider it as the difference between the volumes of a spherical sector and a right circular cone. Compute the volume in Ex. 46 by this method.

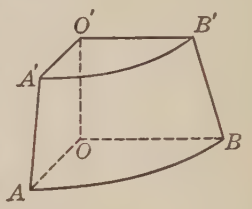
That is, the base of the spherical sector is the zone of one base formed by the circle of which the arc $P'Q'$ shown in the figure is a quadrant.

48. This figure shows a fourth of a spherical sector formed by the revolution of a circular sector of 30° about AO as an axis. Compute the area of the total surface and the volume of the spherical sector when the radius of the circle from which the given circular sector was cut is 7 in.



49. Find the area of the total surface and the volume of a regular octagonal pyramid inscribed in a right circular cone 3.5 in. in diameter and 4.5 in. high.

50. This figure shows one fourth of a frustum of a right circular cone. Find the area of the total surface and the volume of such a frustum when cut from a cone 3.5 in. in diameter and 12 in. high by passing a plane parallel to the base at a distance of 2.5 in.



51. In Ex. 50 the volume of the frustum is what part of the volume of the cone from which it was cut?

52. In Ex. 50 how far above the base should the plane be passed in order to have the volume of the frustum equal to two thirds the volume of the cone?

53. Find the area of the total surface and the volume of a regular pentagonal pyramid inscribed in a circular cone 8 in. in diameter and 10 in. high.

54. A regular octagonal pyramid has a base 1.5 in. on an edge, and each of the lateral edges is 2.5 in. long. Find the diameter and the height of the circular cone which could be circumscribed about this pyramid.

55. A steel container has the shape of a cylinder with a hemispherical top. If the inside length of the cylindric portion is 4 ft. and the diameter is 1 ft. 6 in., what is the volume of the container?

56. Find the diameter of the largest solid sphere that can be formed by melting and recasting the metal contained in two solid lead spheres which are respectively 2 in. and 3 in. in diameter.

57. How many linear feet of brass wire 0.1 in. in diameter can be made by drawing out 1 cu. ft. of brass?

58. Find the least amount of wood that it is necessary to waste in cutting a cube from a wooden sphere which is 5.5 in. in diameter.

TABLES

1. Powers and Roots

Table of Square Roots, 1.00–99.9. The table on pages 352 and 353 gives, to the nearest 0.001, the square roots of numbers from 1.00 to 9.99, and the table on pages 354 and 355 gives the square roots of numbers from 10.0 to 99.9.

In finding the square root of 2.78, for example, we first look in the left-hand column on page 352 for 2.7. Then in the same line, in the column headed 8, we find 1.667; that is, $\sqrt{2.78} = 1.667$, to the nearest 0.001.

Since $\sqrt{100} = 10$ and $\sqrt{0.01} = 0.1$, we can extend the table to cases like finding the value of $\sqrt{278}$ and $\sqrt{0.0278}$.

For example, we have $\sqrt{278} = \sqrt{100 \times 2.78} = 10 \times 1.667 = 16.67$, and $\sqrt{0.0278} = \sqrt{0.01 \times 2.78} = 0.1 \times 1.667 = 0.1667$. In practice we simply look up $\sqrt{2.78}$ and write the decimal point in its proper place.

The columns 1 to 9 at the right of each page are for use in cases where the given number has a fourth figure.

For example, as above, $\sqrt{2.78} = 1.667$. If we wish to find $\sqrt{2.782}$, we look in column 2 at the right on line 2.7 and find 1, which means that 0.001 is to be added to 1.667; that is,

$$\sqrt{2.782} = 1.667 + 0.001 = 1.668.$$

Table of Powers and Roots, 1–100. The table on page 356 gives in a convenient and compact form the squares, cubes, square roots, and cube roots of the integers from 1 to 100. The use of this table is practically self-evident.

For example, from the column headed "Square Roots" we easily see that $\sqrt{33} = 5.745$, while from the column headed "Cube Roots" we see that $\sqrt[3]{74} = 4.198$, each to the nearest 0.001.

In the column headed "Cubes" we also see that $4913 = 17^3$; that is, $\sqrt[3]{4913} = 17$. In finding square roots, the tables on pages 352-355 are generally more convenient.

SQUARE ROOTS. 1.00–5.49

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	1.000	1.005	1.010	1.015	1.020	1.025	1.030	1.034	1.039	1.044	0	1	1	2	2	3	3	4	4
1.1	1.049	1.054	1.058	1.063	1.068	1.072	1.077	1.082	1.086	1.091	0	1	1	2	2	3	3	4	4
1.2	1.095	1.100	1.105	1.109	1.114	1.118	1.122	1.127	1.131	1.136	0	1	1	2	2	3	3	4	4
1.3	1.140	1.145	1.149	1.153	1.158	1.162	1.166	1.170	1.175	1.179	0	1	1	2	2	3	3	4	
1.4	1.183	1.187	1.192	1.196	1.200	1.204	1.208	1.212	1.217	1.221	0	1	1	2	2	2	3	3	4
1.5	1.225	1.229	1.233	1.237	1.241	1.245	1.249	1.253	1.257	1.261	0	1	1	2	2	2	3	3	4
1.6	1.265	1.269	1.273	1.277	1.281	1.285	1.288	1.292	1.296	1.300	0	1	1	2	2	2	3	3	4
1.7	1.304	1.308	1.311	1.315	1.319	1.323	1.327	1.330	1.334	1.338	0	1	1	2	2	2	3	3	3
1.8	1.342	1.345	1.349	1.353	1.356	1.360	1.364	1.367	1.371	1.375	0	1	1	1	2	2	3	3	3
1.9	1.378	1.382	1.386	1.389	1.393	1.396	1.400	1.404	1.407	1.411	0	1	1	1	2	2	3	3	3
2.0	1.414	1.418	1.421	1.425	1.428	1.432	1.435	1.439	1.442	1.446	0	1	1	1	2	2	2	3	3
2.1	1.449	1.453	1.456	1.459	1.463	1.466	1.470	1.473	1.476	1.480	0	1	1	1	2	2	2	3	3
2.2	1.483	1.487	1.490	1.493	1.497	1.500	1.503	1.507	1.510	1.513	0	1	1	1	2	2	2	3	3
2.3	1.517	1.520	1.523	1.526	1.530	1.533	1.536	1.539	1.543	1.546	0	1	1	1	2	2	2	3	3
2.4	1.549	1.552	1.556	1.559	1.562	1.565	1.568	1.572	1.575	1.578	0	1	1	1	2	2	2	3	3
2.5	1.581	1.584	1.587	1.591	1.594	1.597	1.600	1.603	1.606	1.609	0	1	1	1	2	2	2	3	3
2.6	1.612	1.616	1.619	1.622	1.625	1.628	1.631	1.634	1.637	1.640	0	1	1	1	2	2	2	3	3
2.7	1.643	1.646	1.649	1.652	1.655	1.658	1.661	1.664	1.667	1.670	0	1	1	1	2	2	2	2	3
2.8	1.673	1.676	1.679	1.682	1.685	1.688	1.691	1.694	1.697	1.700	0	1	1	1	1	2	2	2	3
2.9	1.703	1.706	1.709	1.712	1.715	1.718	1.720	1.723	1.726	1.729	0	1	1	1	1	2	2	2	3
3.0	1.732	1.735	1.738	1.741	1.744	1.746	1.749	1.752	1.755	1.758	0	1	1	1	1	2	2	2	3
3.1	1.761	1.764	1.766	1.769	1.772	1.775	1.778	1.780	1.783	1.786	0	1	1	1	1	2	2	2	3
3.2	1.789	1.792	1.794	1.797	1.800	1.803	1.806	1.808	1.811	1.814	0	1	1	1	1	2	2	2	2
3.3	1.817	1.819	1.822	1.825	1.828	1.830	1.833	1.836	1.838	1.841	0	1	1	1	1	2	2	2	2
3.4	1.844	1.847	1.849	1.852	1.855	1.857	1.860	1.863	1.865	1.868	0	1	1	1	1	2	2	2	2
3.5	1.871	1.873	1.876	1.879	1.881	1.884	1.887	1.889	1.892	1.895	0	1	1	1	1	2	2	2	2
3.6	1.897	1.900	1.903	1.905	1.908	1.910	1.913	1.916	1.918	1.921	0	1	1	1	1	2	2	2	2
3.7	1.924	1.926	1.929	1.931	1.934	1.936	1.939	1.942	1.944	1.947	0	1	1	1	1	2	2	2	2
3.8	1.949	1.952	1.954	1.957	1.960	1.962	1.965	1.967	1.970	1.972	0	1	1	1	1	2	2	2	2
3.9	1.975	1.977	1.980	1.982	1.985	1.987	1.990	1.992	1.995	1.997	0	1	1	1	1	2	2	2	2
4.0	2.000	2.002	2.005	2.007	2.010	2.012	2.015	2.017	2.020	2.022	0	0	1	1	1	1	2	2	2
4.1	2.025	2.027	2.030	2 032	2.035	2.037	2.040	2.042	2.045	2.047	0	0	1	1	1	1	2	2	2
4.2	2.049	2.052	2.054	2.057	2.059	2.062	2.064	2.066	2.069	2.071	0	0	1	1	1	1	2	2	2
4.3	2.074	2.076	2.078	2.081	2.083	2.086	2.088	2.090	2.093	2.095	0	0	1	1	1	1	2	2	2
4.4	2.098	2.100	2.102	2.105	2.107	2.110	2.112	2.114	2.117	2.119	0	0	1	1	1	1	2	2	2
4.5	2.121	2.124	2.126	2.128	2.131	2.133	2.135	2.138	2.140	2 142	0	0	1	1	1	1	2	2	2
4.6	2.145	2.147	2.149	2.152	2.154	2.156	2.159	2.161	2.163	2.166	0	0	1	1	1	1	2	2	2
4.7	2.168	2.170	2.173	2.175	2.177	2.179	2.182	2.184	2.186	2.189	0	0	1	1	1	1	2	2	2
4.8	2.191	2.193	2.195	2.198	2.200	2.202	2.205	2.207	2.209	2.211	0	0	1	1	1	1	2	2	2
4.9	2.214	2.216	2.218	2.220	2.223	2.225	2.227	2.229	2.232	2.234	0	0	1	1	1	1	2	2	2
5.0	2.236	2.238	2.241	2.243	2.245	2.247	2.249	2.252	2.254	2.256	0	0	1	1	1	1	2	2	2
5.1	2.258	2.261	2.263	2.265	2.267	2.269	2.272	2.274	2.276	2.278	0	0	1	1	1	1	2	2	2
5.2	2.280	2.283	2.285	2.287	2.289	2.291	2.293	2.296	2.298	2.300	0	0	1	1	1	1	2	2	2
5.3	2.302	2.304	2.307	2.309	2.311	2.313	2.315	2.317	2.319	2.322	0	0	1	1	1	1	2	2	2
5.4	2.324	2.326	2.328	2.330	2.332	2.335	2.337	2.339	2.341	2.343	0	0	1	1	1	1	1	2	2

The figures in the difference columns are to be added.

SQUARE ROOTS. 5.50–9.99

N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
5.5	2.345	2.347	2.349	2.352	2.354	2.356	2.358	2.360	2.362	2.364	0 0 1	1 1 1	1 2 2
5.6	2.366	2.369	2.371	2.373	2.375	2.377	2.379	2.381	2.383	2.385	0 0 1	1 1 1	1 2 2
5.7	2.387	2.390	2.392	2.394	2.396	2.398	2.400	2.402	2.404	2.406	0 0 1	1 1 1	1 2 2
5.8	2.408	2.410	2.412	2.415	2.417	2.419	2.421	2.423	2.425	2.427	0 0 1	1 1 1	1 2 2
5.9	2.429	2.431	2.433	2.435	2.437	2.439	2.441	2.443	2.445	2.447	0 0 1	1 1 1	1 2 2
6.0	2.449	2.452	2.454	2.456	2.458	2.460	2.462	2.464	2.466	2.468	0 0 1	1 1 1	1 2 2
6.1	2.470	2.472	2.474	2.476	2.478	2.480	2.482	2.484	2.486	2.488	0 0 1	1 1 1	1 2 2
6.2	2.490	2.492	2.494	2.496	2.498	2.500	2.502	2.504	2.506	2.508	0 0 1	1 1 1	1 2 2
6.3	2.510	2.512	2.514	2.516	2.518	2.520	2.522	2.524	2.526	2.528	0 0 1	1 1 1	1 2 2
6.4	2.530	2.532	2.534	2.536	2.538	2.540	2.542	2.544	2.546	2.548	0 0 1	1 1 1	1 2 2
6.5	2.550	2.551	2.553	2.555	2.557	2.559	2.561	2.563	2.565	2.567	0 0 1	1 1 1	1 2 2
6.6	2.569	2.571	2.573	2.575	2.577	2.579	2.581	2.583	2.585	2.587	0 0 1	1 1 1	1 2 2
6.7	2.588	2.590	2.592	2.594	2.596	2.598	2.600	2.602	2.604	2.606	0 0 1	1 1 1	1 2 2
6.8	2.608	2.610	2.612	2.613	2.615	2.617	2.619	2.621	2.623	2.625	0 0 1	1 1 1	1 2 2
6.9	2.627	2.629	2.631	2.632	2.634	2.636	2.638	2.640	2.642	2.644	0 0 1	1 1 1	1 2 2
7.0	2.646	2.648	2.650	2.651	2.653	2.655	2.657	2.659	2.661	2.663	0 0 1	1 1 1	1 2 2
7.1	2.665	2.666	2.668	2.670	2.672	2.674	2.676	2.678	2.680	2.681	0 0 1	1 1 1	1 1 2
7.2	2.683	2.685	2.687	2.689	2.691	2.693	2.694	2.696	2.698	2.700	0 0 1	1 1 1	1 1 2
7.3	2.702	2.704	2.706	2.707	2.709	2.711	2.713	2.715	2.717	2.718	0 0 1	1 1 1	1 1 2
7.4	2.720	2.722	2.724	2.726	2.728	2.729	2.731	2.733	2.735	2.737	0 0 1	1 1 1	1 1 2
7.5	2.739	2.740	2.742	2.744	2.746	2.748	2.750	2.751	2.753	2.755	0 0 1	1 1 1	1 1 2
7.6	2.757	2.759	2.760	2.762	2.764	2.766	2.768	2.769	2.771	2.773	0 0 1	1 1 1	1 1 2
7.7	2.775	2.777	2.778	2.780	2.782	2.784	2.786	2.787	2.789	2.791	0 0 1	1 1 1	1 1 2
7.8	2.793	2.795	2.796	2.798	2.800	2.802	2.804	2.805	2.807	2.809	0 0 1	1 1 1	1 1 2
7.9	2.811	2.812	2.814	2.816	2.818	2.820	2.821	2.823	2.825	2.827	0 0 1	1 1 1	1 1 2
8.0	2.828	2.830	2.832	2.834	2.835	2.837	2.839	2.841	2.843	2.844	0 0 1	1 1 1	1 1 2
8.1	2.846	2.848	2.850	2.851	2.853	2.855	2.857	2.858	2.860	2.862	0 0 1	1 1 1	1 1 2
8.2	2.864	2.865	2.867	2.869	2.871	2.872	2.874	2.876	2.877	2.879	0 0 1	1 1 1	1 1 2
8.3	2.881	2.883	2.884	2.886	2.888	2.890	2.891	2.893	2.895	2.897	0 0 1	1 1 1	1 1 2
8.4	2.898	2.900	2.902	2.903	2.905	2.907	2.909	2.910	2.912	2.914	0 0 1	1 1 1	1 1 2
8.5	2.915	2.917	2.919	2.921	2.922	2.924	2.926	2.927	2.929	2.931	0 0 1	1 1 1	1 1 2
8.6	2.933	2.934	2.936	2.938	2.939	2.941	2.943	2.944	2.946	2.948	0 0 1	1 1 1	1 1 2
8.7	2.950	2.951	2.953	2.955	2.956	2.958	2.960	2.961	2.963	2.965	0 0 1	1 1 1	1 1 2
8.8	2.966	2.968	2.970	2.972	2.973	2.975	2.977	2.978	2.980	2.982	0 0 1	1 1 1	1 1 2
8.9	2.983	2.985	2.987	2.988	2.990	2.992	2.993	2.995	2.997	2.998	0 0 1	1 1 1	1 1 2
9.0	3.000	3.002	3.003	3.005	3.007	3.008	3.010	3.012	3.013	3.015	0 0 0	1 1 1	1 1 1
9.1	3.017	3.018	3.020	3.022	3.023	3.025	3.027	3.028	3.030	3.032	0 0 0	1 1 1	1 1 1
9.2	3.033	3.035	3.036	3.038	3.040	3.041	3.043	3.045	3.046	3.048	0 0 0	1 1 1	1 1 1
9.3	3.050	3.051	3.053	3.055	3.056	3.058	3.059	3.061	3.063	3.064	0 0 0	1 1 1	1 1 1
9.4	3.066	3.068	3.069	3.071	3.072	3.074	3.076	3.077	3.079	3.081	0 0 0	1 1 1	1 1 1
9.5	3.082	3.084	3.085	3.087	3.089	3.090	3.092	3.094	3.095	3.097	0 0 0	1 1 1	1 1 1
9.6	3.098	3.100	3.102	3.103	3.105	3.106	3.108	3.110	3.111	3.113	0 0 0	1 1 1	1 1 1
9.7	3.114	3.116	3.118	3.119	3.121	3.122	3.124	3.126	3.127	3.129	0 0 0	1 1 1	1 1 1
9.8	3.130	3.132	3.134	3.135	3.137	3.138	3.140	3.142	3.143	3.145	0 0 0	1 1 1	1 1 1
9.9	3.146	3.148	3.150	3.151	3.153	3.154	3.156	3.158	3.159	3.161	0 0 0	1 1 1	1 1 1

The figures in the difference columns are to be added.

SQUARE ROOTS. 10.0–54.9

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	3.162	3.178	3.194	3.209	3.225	3.240	3.256	3.271	3.286	3.302	2	3	5	6	8	9	11	12	14
11	3.317	3.332	3.347	3.362	3.376	3.391	3.406	3.421	3.435	3.450	1	3	4	6	7	9	10	12	13
12	3.464	3.479	3.493	3.507	3.521	3.536	3.550	3.564	3.578	3.592	1	3	4	6	7	8	10	11	13
13	3.606	3.619	3.633	3.647	3.661	3.674	3.688	3.701	3.715	3.728	1	3	4	5	7	8	10	11	12
14	3.742	3.755	3.768	3.782	3.795	3.808	3.821	3.834	3.847	3.860	1	3	4	5	7	8	9	11	12
15	3.873	3.886	3.899	3.912	3.924	3.937	3.950	3.962	3.975	3.987	1	3	4	5	6	8	9	10	11
16	4.000	4.012	4.025	4.037	4.050	4.062	4.074	4.087	4.099	4.111	1	2	4	5	6	7	9	10	11
17	4.123	4.135	4.147	4.159	4.171	4.183	4.195	4.207	4.219	4.231	1	2	4	5	6	7	8	10	11
18	4.243	4.254	4.266	4.278	4.290	4.301	4.313	4.324	4.336	4.347	1	2	3	5	6	7	8	9	10
19	4.359	4.370	4.382	4.393	4.405	4.416	4.427	4.438	4.450	4.461	1	2	3	5	6	7	8	9	10
20	4.472	4.483	4.494	4.506	4.517	4.528	4.539	4.550	4.561	4.572	1	2	3	4	6	7	8	9	10
21	4.583	4.593	4.604	4.615	4.626	4.637	4.648	4.658	4.669	4.680	1	2	3	4	5	6	8	9	10
22	4.690	4.701	4.712	4.722	4.733	4.743	4.754	4.764	4.775	4.785	1	2	3	4	5	6	7	8	9
23	4.796	4.806	4.817	4.827	4.837	4.848	4.858	4.868	4.879	4.889	1	2	3	4	5	6	7	8	9
24	4.899	4.909	4.919	4.930	4.940	4.950	4.960	4.970	4.980	4.990	1	2	3	4	5	6	7	8	9
25	5.000	5.010	5.020	5.030	5.040	5.050	5.060	5.070	5.079	5.089	1	2	3	4	5	6	7	8	9
26	5.099	5.109	5.119	5.128	5.138	5.148	5.158	5.167	5.177	5.187	1	2	3	4	5	6	7	8	9
27	5.196	5.206	5.215	5.225	5.235	5.244	5.254	5.263	5.273	5.282	1	2	3	4	5	6	7	8	9
28	5.292	5.301	5.310	5.320	5.329	5.339	5.348	5.357	5.367	5.376	1	2	3	4	5	6	7	7	8
29	5.385	5.394	5.404	5.413	5.422	5.431	5.441	5.450	5.459	5.468	1	2	3	4	5	5	6	7	8
30	5.477	5.486	5.495	5.505	5.514	5.523	5.532	5.541	5.550	5.559	1	2	3	4	4	5	6	7	8
31	5.568	5.577	5.586	5.595	5.604	5.612	5.621	5.630	5.639	5.648	1	2	3	3	4	5	6	7	8
32	5.657	5.666	5.675	5.683	5.692	5.701	5.710	5.718	5.727	5.736	1	2	3	3	4	5	6	7	8
33	5.745	5.753	5.762	5.771	5.779	5.788	5.797	5.805	5.814	5.822	1	2	3	3	4	5	6	7	8
34	5.831	5.840	5.848	5.857	5.865	5.874	5.882	5.891	5.899	5.908	1	2	3	3	4	5	6	7	8
35	5.916	5.925	5.933	5.941	5.950	5.958	5.967	5.975	5.983	5.992	1	2	2	3	4	5	6	7	8
36	6.000	6.008	6.017	6.025	6.033	6.042	6.050	6.058	6.066	6.075	1	2	2	3	4	5	6	7	7
37	6.083	6.091	6.099	6.107	6.116	6.124	6.132	6.140	6.148	6.156	1	2	2	3	4	5	6	7	7
38	6.164	6.173	6.181	6.189	6.197	6.205	6.213	6.221	6.229	6.237	1	2	2	3	4	5	6	6	7
39	6.245	6.253	6.261	6.269	6.277	6.285	6.293	6.301	6.309	6.317	1	2	2	3	4	5	6	6	7
40	6.325	6.332	6.340	6.348	6.356	6.364	6.372	6.380	6.387	6.395	1	2	2	3	4	5	6	6	7
41	6.403	6.411	6.419	6.427	6.434	6.442	6.450	6.458	6.465	6.473	1	2	2	3	4	5	5	6	7
42	6.481	6.488	6.496	6.504	6.512	6.519	6.527	6.535	6.542	6.550	1	2	2	3	4	5	5	6	7
43	6.557	6.565	6.573	6.580	6.588	6.595	6.603	6.611	6.618	6.626	1	2	2	3	4	5	5	6	7
44	6.633	6.641	6.648	6.656	6.663	6.671	6.678	6.686	6.693	6.701	1	2	2	3	4	5	5	6	7
45	6.708	6.716	6.723	6.731	6.738	6.745	6.753	6.760	6.768	6.775	1	1	2	3	4	4	5	6	7
46	6.782	6.790	6.797	6.804	6.812	6.819	6.826	6.834	6.841	6.848	1	1	2	3	4	4	5	6	7
47	6.856	6.863	6.870	6.877	6.885	6.892	6.899	6.907	6.914	6.921	1	1	2	3	4	4	5	6	7
48	6.928	6.935	6.943	6.950	6.957	6.964	6.971	6.979	6.986	6.993	1	1	2	3	4	4	5	6	6
49	7.000	7.007	7.014	7.021	7.029	7.036	7.043	7.050	7.057	7.064	1	1	2	3	4	4	5	6	6
50	7.071	7.078	7.085	7.092	7.099	7.106	7.113	7.120	7.127	7.134	1	1	2	3	4	4	5	6	6
51	7.141	7.148	7.155	7.162	7.169	7.176	7.183	7.190	7.197	7.204	1	1	2	3	4	4	5	6	6
52	7.211	7.218	7.225	7.232	7.239	7.246	7.253	7.259	7.266	7.273	1	1	2	3	3	4	5	6	6
53	7.280	7.287	7.294	7.301	7.308	7.314	7.321	7.328	7.335	7.342	1	1	2	3	3	4	5	5	6
54	7.348	7.355	7.362	7.369	7.376	7.382	7.389	7.396	7.403	7.409	1	1	2	3	3	4	5	5	6

The figures in the difference columns are to be added.

SQUARE ROOTS. 55.0–99.9

N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
55	7.416	7.423	7.430	7.436	7.443	7.450	7.457	7.463	7.470	7.477	1 1 2	3 3 4	5 5 6
56	7.483	7.490	7.497	7.503	7.510	7.517	7.523	7.530	7.537	7.543	1 1 2	3 3 4	5 5 6
57	7.550	7.556	7.563	7.570	7.576	7.583	7.589	7.596	7.603	7.609	1 1 2	3 3 4	5 5 6
58	7.616	7.622	7.629	7.635	7.642	7.649	7.655	7.662	7.668	7.675	1 1 2	3 3 4	5 5 6
59	7.681	7.688	7.694	7.701	7.707	7.714	7.720	7.727	7.733	7.740	1 1 2	3 3 4	4 5 6
60	7.746	7.752	7.759	7.765	7.772	7.778	7.785	7.791	7.797	7.804	1 1 2	3 3 4	4 5 6
61	7.810	7.817	7.823	7.829	7.836	7.842	7.849	7.855	7.861	7.868	1 1 2	3 3 4	4 5 6
62	7.874	7.880	7.887	7.893	7.899	7.906	7.912	7.918	7.925	7.931	1 1 2	3 3 4	4 5 6
63	7.937	7.944	7.950	7.956	7.962	7.969	7.975	7.981	7.987	7.994	1 1 2	3 3 4	4 5 6
64	8.000	8.006	8.012	8.019	8.025	8.031	8.037	8.044	8.050	8.056	1 1 2	2 3 4	4 5 6
65	8.062	8.068	8.075	8.081	8.087	8.093	8.099	8.106	8.112	8.118	1 1 2	2 3 4	4 5 5
66	8.124	8.130	8.136	8.142	8.149	8.155	8.161	8.167	8.173	8.179	1 1 2	2 3 4	4 5 5
67	8.185	8.191	8.198	8.204	8.210	8.216	8.222	8.228	8.234	8.240	1 1 2	2 3 4	4 5 5
68	8.246	8.252	8.258	8.264	8.270	8.276	8.283	8.289	8.295	8.301	1 1 2	2 3 4	4 5 5
69	8.307	8.313	8.319	8.325	8.331	8.337	8.343	8.349	8.355	8.361	1 1 2	2 3 4	4 5 5
70	8.367	8.373	8.379	8.385	8.390	8.396	8.402	8.408	8.414	8.420	1 1 2	2 3 4	4 5 5
71	8.426	8.432	8.438	8.444	8.450	8.456	8.462	8.468	8.473	8.479	1 1 2	2 3 4	4 5 5
72	8.485	8.491	8.497	8.503	8.509	8.515	8.521	8.526	8.532	8.538	1 1 2	2 3 3	4 5 5
73	8.544	8.550	8.556	8.562	8.567	8.573	8.579	8.585	8.591	8.597	1 1 2	2 3 3	4 5 5
74	8.602	8.608	8.614	8.620	8.626	8.631	8.637	8.643	8.649	8.654	1 1 2	2 3 3	4 5 5
75	8.660	8.666	8.672	8.678	8.683	8.689	8.695	8.701	8.706	8.712	1 1 2	2 3 3	4 5 5
76	8.718	8.724	8.729	8.735	8.741	8.746	8.752	8.758	8.764	8.769	1 1 2	2 3 3	4 5 5
77	8.775	8.781	8.786	8.792	8.798	8.803	8.809	8.815	8.820	8.826	1 1 2	2 3 3	4 4 5
78	8.832	8.837	8.843	8.849	8.854	8.860	8.866	8.871	8.877	8.883	1 1 2	2 3 3	4 4 5
79	8.888	8.894	8.899	8.905	8.911	8.916	8.922	8.927	8.933	8.939	1 1 2	2 3 3	4 4 5
80	8.944	8.950	8.955	8.961	8.967	8.972	8.978	8.983	8.989	8.994	1 1 2	2 3 3	4 4 5
81	9.000	9.006	9.011	9.017	9.022	9.028	9.033	9.039	9.044	9.050	1 1 2	2 3 3	4 4 5
82	9.055	9.061	9.066	9.072	9.077	9.083	9.088	9.094	9.099	9.105	1 1 2	2 3 3	4 4 5
83	9.110	9.116	9.121	9.127	9.132	9.138	9.143	9.149	9.154	9.160	1 1 2	2 3 3	4 4 5
84	9.165	9.171	9.176	9.182	9.187	9.192	9.198	9.203	9.209	9.214	1 1 2	2 3 3	4 4 5
85	9.220	9.225	9.230	9.236	9.241	9.247	9.252	9.257	9.263	9.268	1 1 2	2 3 3	4 4 5
86	9.274	9.279	9.284	9.290	9.295	9.301	9.306	9.311	9.317	9.322	1 1 2	2 3 3	4 4 5
87	9.327	9.333	9.338	9.343	9.349	9.354	9.359	9.365	9.370	9.375	1 1 2	2 3 3	4 4 5
88	9.381	9.386	9.391	9.397	9.402	9.407	9.413	9.418	9.423	9.429	1 1 2	2 3 3	4 4 5
89	9.434	9.439	9.445	9.450	9.455	9.460	9.466	9.471	9.476	9.482	1 1 2	2 3 3	4 4 5
90	9.487	9.492	9.497	9.503	9.508	9.513	9.518	9.524	9.529	9.534	1 1 2	2 3 3	4 4 5
91	9.539	9.545	9.550	9.555	9.560	9.566	9.571	9.576	9.581	9.586	1 1 2	2 3 3	4 4 5
92	9.592	9.597	9.602	9.607	9.612	9.618	9.623	9.628	9.633	9.638	1 1 2	2 3 3	4 4 5
93	9.644	9.649	9.654	9.659	9.664	9.670	9.675	9.680	9.685	9.690	1 1 2	2 3 3	4 4 5
94	9.695	9.701	9.706	9.711	9.716	9.721	9.726	9.731	9.737	9.742	1 1 2	2 3 3	4 4 5
95	9.747	9.752	9.757	9.762	9.767	9.772	9.778	9.783	9.788	9.793	1 1 2	2 3 3	4 4 5
96	9.798	9.803	9.808	9.813	9.818	9.823	9.829	9.834	9.839	9.844	1 1 2	2 3 3	4 4 5
97	9.849	9.854	9.859	9.864	9.869	9.874	9.879	9.884	9.889	9.894	1 1 2	2 3 3	4 4 5
98	9.899	9.905	9.910	9.915	9.920	9.925	9.930	9.935	9.940	9.945	0 1 1	2 2 3	3 4 4
99	9.950	9.955	9.960	9.965	9.970	9.975	9.980	9.985	9.990	9.995	0 1 1	2 2 3	3 4 4

The figures in the difference columns are to be added.

POWERS AND ROOTS. 1–100

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2 601	132 651	7.141	3.708
2	4	8	1.414	1.260	52	2 704	140 608	7.211	3.733
3	9	27	1.732	1.442	53	2 809	148 877	7.280	3.756
4	16	64	2.000	1.587	54	2 916	157 464	7.348	3.780
5	25	125	2.236	1.710	55	3 025	166 375	7.416	3.803
6	36	216	2.449	1.817	56	3 136	175 616	7.483	3.826
7	49	343	2.646	1.913	57	3 249	185 193	7.550	3.849
8	64	512	2.828	2.000	58	3 364	195 112	7.616	3.871
9	81	729	3.000	2.080	59	3 481	205 379	7.681	3.893
10	100	1 000	3.162	2.154	60	3 600	216 000	7.746	3.915
11	121	1 331	3.317	2.224	61	3 721	226 981	7.810	3.936
12	144	1 728	3.464	2.289	62	3 844	238 328	7.874	3.958
13	169	2 197	3.606	2.351	63	3 969	250 047	7.937	3.979
14	196	2 744	3.742	2.410	64	4 096	262 144	8.000	4.000
15	225	3 375	3.873	2.466	65	4 225	274 625	8.062	4.021
16	256	4 096	4.000	2.520	66	4 356	287 496	8.124	4.041
17	289	4 913	4.123	2.571	67	4 489	300 763	8.185	4.062
18	324	5 832	4.243	2.621	68	4 624	314 432	8.246	4.082
19	361	6 859	4.359	2.668	69	4 761	328 509	8.307	4.102
20	400	8 000	4.472	2.714	70	4 900	343 000	8.367	4.121
21	441	9 261	4.583	2.759	71	5 041	357 911	8.426	4.141
22	484	10 648	4.690	2.802	72	5 184	373 248	8.485	4.160
23	529	12 167	4.796	2.844	73	5 329	389 017	8.544	4.179
24	576	13 824	4.899	2.884	74	5 476	405 224	8.602	4.198
25	625	15 625	5.000	2.924	75	5 625	421 875	8.660	4.217
26	676	17 576	5.099	2.962	76	5 776	438 976	8.718	4.236
27	729	19 683	5.196	3.000	77	5 929	456 533	8.775	4.254
28	784	21 952	5.292	3.037	78	6 084	474 552	8.832	4.273
29	841	24 389	5.385	3.072	79	6 241	493 039	8.888	4.291
30	900	27 000	5.477	3.107	80	6 400	512 000	8.944	4.309
31	961	29 791	5.568	3.141	81	6 561	531 441	9.000	4.327
32	1 024	32 768	5.657	3.175	82	6 724	551 368	9.055	4.344
33	1 089	35 937	5.745	3.208	83	6 889	571 787	9.110	4.362
34	1 156	39 304	5.831	3.240	84	7 056	592 704	9.165	4.380
35	1 225	42 875	5.916	3.271	85	7 225	614 125	9.220	4.397
36	1 296	46 656	6.000	3.302	86	7 396	636 056	9.274	4.414
37	1 369	50 653	6.083	3.332	87	7 569	658 503	9.327	4.431
38	1 444	54 872	6.164	3.362	88	7 744	681 472	9.381	4.448
39	1 521	59 319	6.245	3.391	89	7 921	704 969	9.434	4.465
40	1 600	64 000	6.325	3.420	90	8 100	729 000	9.487	4.481
41	1 681	68 921	6.403	3.448	91	8 281	753 571	9.539	4.498
42	1 764	74 088	6.481	3.476	92	8 464	778 688	9.592	4.514
43	1 849	79 507	6.557	3.503	93	8 649	804 357	9.644	4.531
44	1 936	85 184	6.633	3.530	94	8 836	830 584	9.695	4.547
45	2 025	91 125	6.708	3.557	95	9 025	857 375	9.747	4.563
46	2 116	97 336	6.782	3.583	96	9 216	884 736	9.798	4.579
47	2 209	103 823	6.856	3.609	97	9 409	912 673	9.849	4.595
48	2 304	110 592	6.928	3.634	98	9 604	941 192	9.899	4.610
49	2 401	117 649	7.000	3.659	99	9 801	970 299	9.950	4.626
50	2 500	125 000	7.071	3.684	100	10 000	1 000 000	10.000	4.642

For the square roots of 1.00–99.9, see pages 352–355.

2. Logarithms of Numbers

Table of Common Logarithms, 1.00–9.99. The table on pages 358 and 359 gives to four places the common logarithms of the numbers 1.00–9.99.

In the table the characteristic, which is zero in each case, is omitted.

The student is already familiar with the use of this table, both in finding the logarithm of a given number, as described on pages 73 and 74, and in finding the antilogarithm of a given logarithm, as described on pages 75 and 76.

The first two figures of the number are given in the column at the left, headed **N**, and the third figure is given in the columns headed **0** to **9** at the right of column **N**. The difference columns at the right, headed **1** to **9** in smaller type, indicate the number to be added in each case when the number whose logarithm is to be found contains a fourth figure. That is, the table can be used in finding directly the logarithm of any number of four figures.

Instead of using the difference columns, the differences may be found by interpolation, as described on pages 72 and 75. Since many tables of logarithms do not contain difference columns, it is desirable that students should be able to apply this method readily.

The characteristic of the logarithm of a number that is not given in the table is easily found by applying the rules given on page 69.

Thus, since $26.5 = 2.65 \times 10^1$, the characteristic of log 26.5 is 1 greater than that of log 2.65; that is, $\log 26.5 = 1.4232$. In the same way, since $0.00265 = 2.65 \times 10^{-3}$, we have $\log 0.00265 = \bar{3}.4232$.

It should be observed that the table gives the actual logarithms of 1.00–9.99, not just the mantissas. This arrangement is much more convenient for rapid work than that found in the older types of tables.

The advantages of this arrangement are more apparent in the tables of logarithmic functions given on pages 374–381.

LOGARITHMS OF NUMBERS. 1.00–5.49

N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
1.0	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374	4 8 12	17 21 25	29 33 37
1.1	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755	4 8 11	15 19 23	26 30 34
1.2	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106	3 7 10	14 17 21	24 28 31
1.3	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430	3 6 10	13 16 19	23 26 29
1.4	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732	3 6 9	12 15 18	21 24 27
1.5	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014	3 6 8	11 14 17	20 22 25
1.6	.2041	.2068	.2095	.2122	.2148	.2175	.2201	.2227	.2253	.2279	3 5 8	11 13 16	18 21 24
1.7	.2304	.2330	.2355	.2380	.2405	.2430	.2455	.2480	.2504	.2529	2 5 7	10 12 15	17 20 22
1.8	.2553	.2577	.2601	.2625	.2648	.2672	.2695	.2718	.2742	.2765	2 5 7	9 12 14	16 19 21
1.9	.2788	.2810	.2833	.2856	.2878	.2900	.2923	.2945	.2967	.2989	2 4 7	9 11 13	16 18 20
2.0	.3010	.3032	.3054	.3075	.3096	.3118	.3139	.3160	.3181	.3201	2 4 6	8 11 13	15 17 19
2.1	.3222	.3243	.3263	.3284	.3304	.3324	.3345	.3365	.3385	.3404	2 4 6	8 10 12	14 16 18
2.2	.3424	.3444	.3464	.3483	.3502	.3522	.3541	.3560	.3579	.3598	2 4 6	8 10 12	14 15 17
2.3	.3617	.3636	.3655	.3674	.3692	.3711	.3729	.3747	.3766	.3784	2 4 6	7 9 11	13 15 17
2.4	.3802	.3820	.3838	.3856	.3874	.3892	.3909	.3927	.3945	.3962	2 4 5	7 9 11	12 14 16
2.5	.3979	.3997	.4014	.4031	.4048	.4065	.4082	.4099	.4116	.4133	2 3 5	7 9 10	12 14 15
2.6	.4150	.4166	.4183	.4200	.4216	.4232	.4249	.4265	.4281	.4298	2 3 5	7 8 10	11 13 15
2.7	.4314	.4330	.4346	.4362	.4378	.4393	.4409	.4425	.4440	.4456	2 3 5	6 8 9	11 13 14
2.8	.4472	.4487	.4502	.4518	.4533	.4548	.4564	.4579	.4594	.4609	2 3 5	6 8 9	11 12 14
2.9	.4624	.4639	.4654	.4669	.4683	.4698	.4713	.4728	.4742	.4757	1 3 4	6 7 9	10 12 13
3.0	.4771	.4786	.4800	.4814	.4829	.4843	.4857	.4871	.4886	.4900	1 3 4	6 7 9	10 11 13
3.1	.4914	.4928	.4942	.4955	.4969	.4983	.4997	.5011	.5024	.5038	1 3 4	6 7 8	10 11 12
3.2	.5051	.5065	.5079	.5092	.5105	.5119	.5132	.5145	.5159	.5172	1 3 4	5 7 8	9 11 12
3.3	.5185	.5198	.5211	.5224	.5237	.5250	.5263	.5276	.5289	.5302	1 3 4	5 6 8	9 10 12
3.4	.5315	.5328	.5340	.5353	.5366	.5378	.5391	.5403	.5416	.5428	1 3 4	5 6 8	9 10 11
3.5	.5441	.5453	.5465	.5478	.5490	.5502	.5514	.5527	.5539	.5551	1 2 4	5 6 7	9 10 11
3.6	.5563	.5575	.5587	.5599	.5611	.5623	.5635	.5647	.5658	.5670	1 2 4	5 6 7	8 10 11
3.7	.5682	.5694	.5705	.5717	.5729	.5740	.5752	.5763	.5775	.5786	1 2 3	5 6 7	8 9 10
3.8	.5798	.5809	.5821	.5832	.5843	.5855	.5866	.5877	.5888	.5899	1 2 3	5 6 7	8 9 10
3.9	.5911	.5922	.5933	.5944	.5955	.5966	.5977	.5988	.5999	.6010	1 2 3	4 5 7	8 9 10
4.0	.6021	.6031	.6042	.6053	.6064	.6075	.6085	.6096	.6107	.6117	1 2 3	4 5 6	8 9 10
4.1	.6128	.6138	.6149	.6160	.6170	.6180	.6191	.6201	.6212	.6222	1 2 3	4 5 6	7 8 9
4.2	.6232	.6243	.6253	.6263	.6274	.6284	.6294	.6304	.6314	.6325	1 2 3	4 5 6	7 8 9
4.3	.6335	.6345	.6355	.6365	.6375	.6385	.6395	.6405	.6415	.6425	1 2 3	4 5 6	7 8 9
4.4	.6435	.6444	.6454	.6464	.6474	.6484	.6493	.6503	.6513	.6522	1 2 3	4 5 6	7 8 9
4.5	.6532	.6542	.6551	.6561	.6571	.6580	.6590	.6599	.6609	.6618	1 2 3	4 5 6	7 8 9
4.6	.6628	.6637	.6646	.6656	.6665	.6675	.6684	.6693	.6702	.6712	1 2 3	4 5 6	7 7 8
4.7	.6721	.6730	.6739	.6749	.6758	.6767	.6776	.6785	.6794	.6803	1 2 3	4 5 5	6 7 8
4.8	.6812	.6821	.6830	.6839	.6848	.6857	.6866	.6875	.6884	.6893	1 2 3	4 4 5	6 7 8
4.9	.6902	.6911	.6920	.6928	.6937	.6946	.6955	.6964	.6972	.6981	1 2 3	4 4 5	6 7 8
5.0	.6990	.6998	.7007	.7016	.7024	.7033	.7042	.7050	.7059	.7067	1 2 3	3 4 5	6 7 8
5.1	.7076	.7084	.7093	.7101	.7110	.7118	.7126	.7135	.7143	.7152	1 2 3	3 4 5	6 7 8
5.2	.7160	.7168	.7177	.7185	.7193	.7202	.7210	.7218	.7226	.7235	1 2 2	3 4 5	6 7 7
5.3	.7243	.7251	.7259	.7267	.7275	.7284	.7292	.7300	.7308	.7316	1 2 2	3 4 5	6 6 7
5.4	.7324	.7332	.7340	.7348	.7356	.7364	.7372	.7380	.7388	.7396	1 2 2	3 4 5	6 6 7

In each of the above logarithms the characteristic is 0.

LOGARITHMS OF NUMBERS. 5.50–9.99

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
5.5	.7404	.7412	.7419	.7427	.7435	.7443	.7451	.7459	.7466	.7474	1	2	2	3	4	5	5	6	7
5.6	.7482	.7490	.7497	.7505	.7513	.7520	.7528	.7536	.7543	.7551	1	2	2	3	4	5	5	6	7
5.7	.7559	.7566	.7574	.7582	.7589	.7597	.7604	.7612	.7619	.7627	1	2	2	3	4	5	5	6	7
5.8	.7634	.7642	.7649	.7657	.7664	.7672	.7679	.7686	.7694	.7701	1	1	2	3	4	4	5	6	7
5.9	.7709	.7716	.7723	.7731	.7738	.7745	.7752	.7760	.7767	.7774	1	1	2	3	4	4	5	6	7
6.0	.7782	.7789	.7796	.7803	.7810	.7818	.7825	.7832	.7839	.7846	1	1	2	3	4	4	5	6	6
6.1	.7853	.7860	.7868	.7875	.7882	.7889	.7896	.7903	.7910	.7917	1	1	2	3	4	4	5	6	6
6.2	.7924	.7931	.7938	.7945	.7952	.7959	.7966	.7973	.7980	.7987	1	1	2	3	3	4	5	6	6
6.3	.7993	.8000	.8007	.8014	.8021	.8028	.8035	.8041	.8048	.8055	1	1	2	3	3	4	5	5	6
6.4	.8062	.8069	.8075	.8082	.8089	.8096	.8102	.8109	.8116	.8122	1	1	2	3	3	4	5	5	6
6.5	.8129	.8136	.8142	.8149	.8156	.8162	.8169	.8176	.8182	.8189	1	1	2	3	3	4	5	5	6
6.6	.8195	.8202	.8209	.8215	.8222	.8228	.8235	.8241	.8248	.8254	1	1	2	3	3	4	5	5	6
6.7	.8261	.8267	.8274	.8280	.8287	.8293	.8299	.8306	.8312	.8319	1	1	2	3	3	4	5	5	6
6.8	.8325	.8331	.8338	.8344	.8351	.8357	.8363	.8370	.8376	.8382	1	1	2	3	3	4	4	5	6
6.9	.8388	.8395	.8401	.8407	.8414	.8420	.8426	.8432	.8439	.8445	1	1	2	2	3	4	4	5	6
7.0	.8451	.8457	.8463	.8470	.8476	.8482	.8488	.8494	.8500	.8506	1	1	2	2	3	4	4	5	6
7.1	.8513	.8519	.8525	.8531	.8537	.8543	.8549	.8555	.8561	.8567	1	1	2	2	3	4	4	5	5
7.2	.8573	.8579	.8585	.8591	.8597	.8603	.8609	.8615	.8621	.8627	1	1	2	2	3	4	4	5	5
7.3	.8633	.8639	.8645	.8651	.8657	.8663	.8669	.8675	.8681	.8686	1	1	2	2	3	4	4	5	5
7.4	.8692	.8698	.8704	.8710	.8716	.8722	.8727	.8733	.8739	.8745	1	1	2	2	3	4	4	5	5
7.5	.8751	.8756	.8762	.8768	.8774	.8779	.8785	.8791	.8797	.8802	1	1	2	2	3	3	4	5	5
7.6	.8808	.8814	.8820	.8825	.8831	.8837	.8842	.8848	.8854	.8859	1	1	2	2	3	3	4	5	5
7.7	.8865	.8871	.8876	.8882	.8887	.8893	.8899	.8904	.8910	.8915	1	1	2	2	3	3	4	4	5
7.8	.8921	.8927	.8932	.8938	.8943	.8949	.8954	.8960	.8965	.8971	1	1	2	2	3	3	4	4	5
7.9	.8976	.8982	.8987	.8993	.8998	.9004	.9009	.9015	.9020	.9025	1	1	2	2	3	3	4	4	5
8.0	.9031	.9036	.9042	.9047	.9053	.9058	.9063	.9069	.9074	.9079	1	1	2	2	3	3	4	4	5
8.1	.9085	.9090	.9096	.9101	.9106	.9112	.9117	.9122	.9128	.9133	1	1	2	2	3	3	4	4	5
8.2	.9138	.9143	.9149	.9154	.9159	.9165	.9170	.9175	.9180	.9186	1	1	2	2	3	3	4	4	5
8.3	.9191	.9196	.9201	.9206	.9212	.9217	.9222	.9227	.9232	.9238	1	1	2	2	3	3	4	4	5
8.4	.9243	.9248	.9253	.9258	.9263	.9269	.9274	.9279	.9284	.9289	1	1	2	2	3	3	4	4	5
8.5	.9294	.9299	.9304	.9309	.9315	.9320	.9325	.9330	.9335	.9340	1	1	2	2	3	3	4	4	5
8.6	.9345	.9350	.9355	.9360	.9365	.9370	.9375	.9380	.9385	.9390	1	1	2	2	3	3	4	4	5
8.7	.9395	.9400	.9405	.9410	.9415	.9420	.9425	.9430	.9435	.9440	0	1	1	2	2	3	3	4	4
8.8	.9445	.9450	.9455	.9460	.9465	.9469	.9474	.9479	.9484	.9489	0	1	1	2	2	3	3	4	4
8.9	.9494	.9499	.9504	.9509	.9513	.9518	.9523	.9528	.9533	.9538	0	1	1	2	2	3	3	4	4
9.0	.9542	.9547	.9552	.9557	.9562	.9566	.9571	.9576	.9581	.9586	0	1	1	2	2	3	3	4	4
9.1	.9590	.9595	.9600	.9605	.9609	.9614	.9619	.9624	.9628	.9633	0	1	1	2	2	3	3	4	4
9.2	.9638	.9643	.9647	.9652	.9657	.9661	.9666	.9671	.9675	.9680	0	1	1	2	2	3	3	4	4
9.3	.9685	.9689	.9694	.9699	.9703	.9708	.9713	.9717	.9722	.9727	0	1	1	2	2	3	3	4	4
9.4	.9731	.9736	.9741	.9745	.9750	.9754	.9759	.9763	.9768	.9773	0	1	1	2	2	3	3	4	4
9.5	.9777	.9782	.9786	.9791	.9795	.9800	.9805	.9809	.9814	.9818	0	1	1	2	2	3	3	4	4
9.6	.9823	.9827	.9832	.9836	.9841	.9845	.9850	.9854	.9859	.9863	0	1	1	2	2	3	3	4	4
9.7	.9868	.9872	.9877	.9881	.9886	.9890	.9894	.9899	.9903	.9908	0	1	1	2	2	3	3	4	4
9.8	.9912	.9917	.9921	.9926	.9930	.9934	.9939	.9943	.9948	.9952	0	1	1	2	2	3	3	4	4
9.9	.9956	.9961	.9965	.9969	.9974	.9978	.9983	.9987	.9991	.9996	0	1	1	2	2	3	3	3	4

In each of the above logarithms the characteristic is 0.

3. Trigonometric Functions

Natural Functions. On pages 362–373 are given four-place tables of the six important trigonometric functions of an angle, — the sine, cosine, tangent, cotangent, secant, and cosecant. These tables, which are designated as natural functions in order to distinguish them from the logarithmic functions considered later, give the functions for each 0.1°, or 6′, from 0° to 90°. By means of the difference columns at the right of each table, or by using ordinary interpolation, the tables are easily extended to give the functions of an angle measured to the nearest 1′.

In cases requiring greater accuracy, tables with a larger number of places are used, but for ordinary work, such as is emphasized in this book, the tables given here are sufficiently accurate.

The methods of using the tables of functions are already familiar to the student, not only in finding the function of a given angle between 0° and 90° but also in finding the angle between 0° and 90° which has a given function, as described on pages 117–119.

The method of using the difference columns is described on page 119, while the method of interpolation used in cases like finding the tangent of an angle larger than 75° is described on page 118.

In using the tables to find the functions of angles greater than 90°, it is necessary to apply the reduction formulas given on pages 113–115. By means of these formulas the function of the angle is easily reduced to the function of an angle in the first quadrant.

For example, in finding sin 158°, we have

$$\sin 158° = \sin(180° - 22°) = \sin 22° = 0.3746,$$

or $$\sin 158° = \sin(90° + 68°) = \cos 68° = 0.3746.$$

Similarly, $$\cos 300° = \cos(360° - 60°) = \cos 60° = 0.5000,$$

or $$\cos 300° = \cos(270° + 30°) = \sin 30° = 0.5000.$$

Also, $$\tan 250° = \tan(180° + 70°) = \tan 70° = 2.7475,$$

or $$\tan 250° = \tan(270° - 20°) = \cot 20° = 2.7475.$$

Logarithmic Functions. On pages 374–381 are given four-place tables of the logarithms of the sines, cosines, tangents, and cotangents of angles from 0° to 90° for each 0.1°, or 6′.

The method of using these tables is a combination, as described on page 121, of the methods used in connection with a table of logarithms and those used with the tables of natural functions.

It will be observed that no tables are given for the logarithms of the secant and cosecant. These functions, as was seen in Chapter IV, are relatively little used, and in the few cases where we need to use them we may apply the relations $\sec \phi = \frac{1}{\cos \phi}$ and $\csc \phi = \frac{1}{\sin \phi}$.

That is, in multiplying a number by sec ϕ we may do so by subtracting the logarithm of cos ϕ, or we may add the cologarithm of cos ϕ (see page 80), according to which is the more convenient.

The same is also true of cot ϕ; that is, we might apply the relation $\cot \phi = \frac{1}{\tan \phi}$. The cotangent, however, is more often used, and for greater convenience a table of logarithmic cotangents has been included in this group.

It should also be observed that, as mentioned on page 357, the tables of logarithmic functions give the actual logarithms, not simply the mantissas. The greater convenience of this arrangement is seen in those parts of the tables in which the characteristics of the logarithms change frequently.

Furthermore, the correct characteristic is given in each line of the table. In many tables the characteristic of the logarithms are printed 10 too large on account of the difficulty of printing negative characteristics; that is, log tan 55° 30′ is given as 10.1629, and it is understood that 10 is to be subtracted. In the tables in this book a special type has been devised which makes it unnecessary to resort to such a plan. Attention should be called to this point in case students are required to use other tables in examinations.

In finding the logarithmic functions of obtuse angles the reduction formulas are used as described on page 360.

NATURAL SINES. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	.0000	.0017	.0035	.0052	.0070	.0087	.0105	.0122	.0140	.0157	3	6	9	12	15
1	.0175	.0192	.0209	.0227	.0244	.0262	.0279	.0297	.0314	.0332	3	6	9	12	15
2	.0349	.0366	.0384	.0401	.0419	.0436	.0454	.0471	.0488	.0506	3	6	9	12	15
3	.0523	.0541	.0558	.0576	.0593	.0610	.0628	.0645	.0663	.0680	3	6	9	12	15
4	.0698	.0715	.0732	.0750	.0767	.0785	.0802	.0819	.0837	.0854	3	6	9	12	14
5	.0872	.0889	.0906	.0924	.0941	.0958	.0976	.0993	.1011	.1028	3	6	9	12	14
6	.1045	.1063	.1080	.1097	.1115	.1132	.1149	.1167	.1184	.1201	3	6	9	12	14
7	.1219	.1236	.1253	.1271	.1288	.1305	.1323	.1340	.1357	.1374	3	6	9	12	14
8	.1392	.1409	.1426	.1444	.1461	.1478	.1495	.1513	.1530	.1547	3	6	9	12	14
9	.1564	.1582	.1599	.1616	.1633	.1650	.1668	.1685	.1702	.1719	3	6	9	12	14
10	.1736	.1754	.1771	.1788	.1805	.1822	.1840	.1857	.1874	.1891	3	6	9	11	14
11	.1908	.1925	.1942	.1959	.1977	.1994	.2011	.2028	.2045	.2062	3	6	9	11	14
12	.2079	.2096	.2113	.2130	.2147	.2164	.2181	.2198	.2215	.2233	3	6	9	11	14
13	.2250	.2267	.2284	.2300	.2317	.2334	.2351	.2368	.2385	.2402	3	6	8	11	14
14	.2419	.2436	.2453	.2470	.2487	.2504	.2521	.2538	.2554	.2571	3	6	8	11	14
15	.2588	.2605	.2622	.2639	.2656	.2672	.2689	.2706	.2723	.2740	3	6	8	11	14
16	.2756	.2773	.2790	.2807	.2823	.2840	.2857	.2874	.2890	.2907	3	6	8	11	14
17	.2924	.2940	.2957	.2974	.2990	.3007	.3024	.3040	.3057	.3074	3	6	8	11	14
18	.3090	.3107	.3123	.3140	.3156	.3173	.3190	.3206	.3223	.3239	3	6	8	11	14
19	.3256	.3272	.3289	.3305	.3322	.3338	.3355	.3371	.3387	.3404	3	5	8	11	14
20	.3420	.3437	.3453	.3469	.3486	.3502	.3518	.3535	.3551	.3567	3	5	8	11	14
21	.3584	.3600	.3616	.3633	.3649	.3665	.3681	.3697	.3714	.3730	3	5	8	11	14
22	.3746	.3762	.3778	.3795	.3811	.3827	.3843	.3859	.3875	.3891	3	5	8	11	14
23	.3907	.3923	.3939	.3955	.3971	.3987	.4003	.4019	.4035	.4051	3	5	8	11	14
24	.4067	.4083	.4099	.4115	.4131	.4147	.4163	.4179	.4195	.4210	3	5	8	11	13
25	.4226	.4242	.4258	.4274	.4289	.4305	.4321	.4337	.4352	.4368	3	5	8	11	13
26	.4384	.4399	.4415	.4431	.4446	.4462	.4478	.4493	.4509	.4524	3	5	8	10	13
27	.4540	.4555	.4571	.4586	.4602	.4617	.4633	.4648	.4664	.4679	3	5	8	10	13
28	.4695	.4710	.4726	.4741	.4756	.4772	.4787	.4802	.4818	.4833	3	5	8	10	13
29	.4848	.4863	.4879	.4894	.4909	.4924	.4939	.4955	.4970	.4985	3	5	8	10	13
30	.5000	.5015	.5030	.5045	.5060	.5075	.5090	.5105	.5120	.5135	3	5	8	10	13
31	.5150	.5165	.5180	.5195	.5210	.5225	.5240	.5255	.5270	.5284	2	5	7	10	12
32	.5299	.5314	.5329	.5344	.5358	.5373	.5388	.5402	.5417	.5432	2	5	7	10	12
33	.5446	.5461	.5476	.5490	.5505	.5519	.5534	.5548	.5563	.5577	2	5	7	10	12
34	.5592	.5606	.5621	.5635	.5650	.5664	.5678	.5693	.5707	.5721	2	5	7	10	12
35	.5736	.5750	.5764	.5779	.5793	.5807	.5821	.5835	.5850	.5864	2	5	7	9	12
36	.5878	.5892	.5906	.5920	.5934	.5948	.5962	.5976	.5990	.6004	2	5	7	9	12
37	.6018	.6032	.6046	.6060	.6074	.6088	.6101	.6115	.6129	.6143	2	5	7	9	12
38	.6157	.6170	.6184	.6198	.6211	.6225	.6239	.6252	.6266	.6280	2	5	7	9	11
39	.6293	.6307	.6320	.6334	.6347	.6361	.6374	.6388	.6401	.6414	2	4	7	9	11
40	.6428	.6441	.6455	.6468	.6481	.6494	.6508	.6521	.6534	.6547	2	4	7	9	11
41	.6561	.6574	.6587	.6600	.6613	.6626	.6639	.6652	.6665	.6678	2	4	7	9	11
42	.6691	.6704	.6717	.6730	.6743	.6756	.6769	.6782	.6794	.6807	2	4	6	9	11
43	.6820	.6833	.6845	.6858	.6871	.6884	.6896	.6909	.6921	.6934	2	4	6	8	11
44	.6947	.6959	.6972	.6984	.6997	.7009	.7022	.7034	.7046	.7059	2	4	6	8	10

All the above sines are less than 1.

NATURAL SINES. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	.7071	.7083	.7096	.7108	.7120	.7133	.7145	.7157	.7169	.7181	2	4	6	8	10
46	.7193	.7206	.7218	.7230	.7242	.7254	.7266	.7278	.7290	.7302	2	4	6	8	10
47	.7314	.7325	.7337	.7349	.7361	.7373	.7385	.7396	.7408	.7420	2	4	6	8	10
48	.7431	.7443	.7455	.7466	.7478	.7490	.7501	.7513	.7524	.7536	2	4	6	8	10
49	.7547	.7559	.7570	.7581	.7593	.7604	.7615	.7627	.7638	.7649	2	4	6	8	9
50	.7660	.7672	.7683	.7694	.7705	.7716	.7727	.7738	.7749	.7760	2	4	6	7	9
51	.7771	.7782	.7793	.7804	.7815	.7826	.7837	.7848	.7859	.7869	2	4	5	7	9
52	.7880	.7891	.7902	.7912	.7923	.7934	.7944	.7955	.7965	.7976	2	4	5	7	9
53	.7986	.7997	.8007	.8018	.8028	.8039	.8049	.8059	.8070	.8080	2	3	5	7	9
54	.8090	.8100	.8111	.8121	.8131	.8141	.8151	.8161	.8171	.8181	2	3	5	7	8
55	.8192	.8202	.8211	.8221	.8231	.8241	.8251	.8261	.8271	.8281	2	3	5	7	8
56	.8290	.8300	.8310	.8320	.8329	.8339	.8348	.8358	.8368	.8377	2	3	5	6	8
57	.8387	.8396	.8406	.8415	.8425	.8434	.8443	.8453	.8462	.8471	2	3	5	6	8
58	.8480	.8490	.8499	.8508	.8517	.8526	.8536	.8545	.8554	.8563	2	3	5	6	8
59	.8572	.8581	.8590	.8599	.8607	.8616	.8625	.8634	.8643	.8652	1	3	4	6	7
60	.8660	.8669	.8678	.8686	.8695	.8704	.8712	.8721	.8729	.8738	1	3	4	6	7
61	.8746	.8755	.8763	.8771	.8780	.8788	.8796	.8805	.8813	.8821	1	3	4	6	7
62	.8829	.8838	.8846	.8854	.8862	.8870	.8878	.8886	.8894	.8902	1	3	4	5	7
63	.8910	.8918	.8926	.8934	.8942	.8949	.8957	.8965	.8973	.8980	1	3	4	5	6
64	.8988	.8996	.9003	.9011	.9018	.9026	.9033	.9041	.9048	.9056	1	3	4	5	6
65	.9063	.9070	.9078	.9085	.9092	.9100	.9107	.9114	.9121	.9128	1	2	4	5	6
66	.9135	.9143	.9150	.9157	.9164	.9171	.9178	.9184	.9191	.9198	1	2	3	5	6
67	.9205	.9212	.9219	.9225	.9232	.9239	.9245	.9252	.9259	.9265	1	2	3	4	6
68	.9272	.9278	.9285	.9291	.9298	.9304	.9311	.9317	.9323	.9330	1	2	3	4	5
69	.9336	.9342	.9348	.9354	.9361	.9367	.9373	.9379	.9385	.9391	1	2	3	4	5
70	.9397	.9403	.9409	.9415	.9421	.9426	.9432	.9438	.9444	.9449	1	2	3	4	5
71	.9455	.9461	.9466	.9472	.9478	.9483	.9489	.9494	.9500	.9505	1	2	3	4	5
72	.9511	.9516	.9521	.9527	.9532	.9537	.9542	.9548	.9553	.9558	1	2	3	4	4
73	.9563	.9568	.9573	.9578	.9583	.9588	.9593	.9598	.9603	.9608	1	2	2	3	4
74	.9613	.9617	.9622	.9627	.9632	.9636	.9641	.9646	.9650	.9655	1	2	2	3	4
75	.9659	.9664	.9668	.9673	.9677	.9681	.9686	.9690	.9694	.9699	1	1	2	3	4
76	.9703	.9707	.9711	.9715	.9720	.9724	.9728	.9732	.9736	.9740	1	1	2	3	3
77	.9744	.9748	.9751	.9755	.9759	.9763	.9767	.9770	.9774	.9778	1	1	2	3	3
78	.9781	.9785	.9789	.9792	.9796	.9799	.9803	.9806	.9810	.9813	1	1	2	2	3
79	.9816	.9820	.9823	.9826	.9829	.9833	.9836	.9839	.9842	.9845	1	1	2	2	3
80	.9848	.9851	.9854	.9857	.9860	.9863	.9866	.9869	.9871	.9874	0	1	1	2	2
81	.9877	.9880	.9882	.9885	.9888	.9890	.9893	.9895	.9898	.9900	0	1	1	2	2
82	.9903	.9905	.9907	.9910	.9912	.9914	.9917	.9919	.9921	.9923	0	1	1	2	2
83	.9925	.9928	.9930	.9932	.9934	.9936	.9938	.9940	.9942	.9943	0	1	1	1	2
84	.9945	.9947	.9949	.9951	.9952	.9954	.9956	.9957	.9959	.9960	0	1	1	1	1
85	.9962	.9963	.9965	.9966	.9968	.9969	.9971	.9972	.9973	.9974	0	0	1	1	1
86	.9976	.9977	.9978	.9979	.9980	.9981	.9982	.9983	.9984	.9985	0	0	1	1	1
87	.9986	.9987	.9988	.9989	.9990	.9990	.9991	.9992	.9993	.9993	0	0	0	1	1
88	.9994	.9995	.9995	.9996	.9996	.9997	.9997	.9997	.9998	.9998	0	0	0	0	0
89	.9998	.9999	.9999	.9999	.9999	1.000	1.000	1.000	1.000	1.000	0	0	0	0	0

The precise value of all sines except sin 90° is less than 1.

NATURAL COSINES. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	— Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	1.000	1.000	1.000	1.000	1.000	1.000	.9999	.9999	.9999	.9999	0	0	0	0	0
1	.9998	.9998	.9998	.9997	.9997	.9997	.9996	.9996	.9995	.9995	0	0	0	0	0
2	.9994	.9993	.9993	.9992	.9991	.9990	.9990	.9989	.9988	.9987	0	0	0	0	0
3	.9986	.9985	.9984	.9983	.9982	.9981	.9980	.9979	.9978	.9977	0	0	1	1	1
4	.9976	.9974	.9973	.9972	.9971	.9969	.9968	.9966	.9965	.9963	0	0	1	1	1
5	.9962	.9960	.9959	.9957	.9956	.9954	.9952	.9951	.9949	.9947	0	1	1	1	1
6	.9945	.9943	.9942	.9940	.9938	.9936	.9934	.9932	.9930	.9928	0	1	1	1	2
7	.9925	.9923	.9921	.9919	.9917	.9914	.9912	.9910	.9907	.9905	0	1	1	2	2
8	.9903	.9900	.9898	.9895	.9893	.9890	.9888	.9885	.9882	.9880	0	1	1	2	2
9	.9877	.9874	.9871	.9869	.9866	.9863	.9860	.9857	.9854	.9851	0	1	1	2	2
10	.9848	.9845	.9842	.9839	.9836	.9833	.9829	.9826	.9823	.9820	1	1	2	2	3
11	.9816	.9813	.9810	.9806	.9803	.9799	.9796	.9792	.9789	.9785	1	1	2	2	3
12	.9781	.9778	.9774	.9770	.9767	.9763	.9759	.9755	.9751	.9748	1	1	2	3	3
13	.9744	.9740	.9736	.9732	.9728	.9724	.9720	.9715	.9711	.9707	1	1	2	3	3
14	.9703	.9699	.9694	.9690	.9686	.9681	.9677	.9673	.9668	.9664	1	1	2	3	4
15	.9659	.9655	.9650	.9646	.9641	9636	.9632	.9627	.9622	.9617	1	2	2	3	4
16	.9613	.9608	.9603	.9598	.9593	.9588	.9583	.9578	.9573	.9568	1	2	2	3	4
17	.9563	.9558	.9553	.9548	.9542	.9537	.9532	.9527	.9521	.9516	1	2	3	4	4
18	.9511	.9505	.9500	.9494	.9489	.9483	.9478	.9472	.9466	.9461	1	2	3	4	5
19	.9455	.9449	.9444	.9438	.9432	.9426	.9421	.9415	.9409	.9403	1	2	3	4	5
20	.9397	.9391	.9385	.9379	.9373	.9367	.9361	.9354	.9348	.9342	1	2	3	4	5
21	.9336	.9330	.9323	.9317	.9311	.9304	.9298	.9291	.9285	.9278	1	2	3	4	5
22	.9272	.9265	.9259	.9252	.9245	.9239	.9232	.9225	.9219	.9212	1	2	3	4	6
23	.9205	.9198	.9191	.9184	.9178	.9171	.9164	.9157	.9150	.9143	1	2	3	5	6
24	.9135	.9128	.9121	.9114	.9107	.9100	.9092	.9085	.9078	.9070	1	2	4	5	6
25	.9063	.9056	.9048	.9041	.9033	.9026	.9018	.9011	.9003	.8996	1	3	4	5	6
26	.8988	.8980	.8973	.8965	.8957	.8949	.8942	.8934	.8926	.8918	1	3	4	5	6
27	.8910	.8902	.8894	.8886	.8878	.8870	.8862	.8854	.8846	.8838	1	3	4	5	7
28	.8829	.8821	.8813	.8805	.8796	.8788	.8780	.8771	.8763	.8755	1	3	4	6	7
29	.8746	.8738	.8729	.8721	.8712	.8704	.8695	.8686	.8678	.8669	1	3	4	6	7
30	.8660	.8652	.8643	.8634	.8625	.8616	.8607	.8599	.8590	.8581	1	3	4	6	7
31	.8572	.8563	.8554	.8545	.8536	.8526	.8517	.8508	.8499	.8490	2	3	5	6	8
32	.8480	.8471	.8462	.8453	.8443	.8434	.8425	.8415	.8406	.8396	2	3	5	6	8
33	.8387	.8377	.8368	.8358	.8348	.8339	.8329	.8320	.8310	.8300	2	3	5	6	8
34	.8290	.8281	.8271	.8261	.8251	.8241	.8231	.8221	.8211	.8202	2	3	5	7	8
35	.8192	.8181	.8171	.8161	.8151	.8141	.8131	.8121	.8111	.8100	2	3	5	7	8
36	.8090	.8080	.8070	.8059	.8049	.8039	.8028	.8018	.8007	.7997	2	3	5	7	9
37	.7986	.7976	.7965	.7955	.7944	.7934	.7923	.7912	.7902	.7891	2	4	5	7	9
38	.7880	.7869	.7859	.7848	.7837	.7826	.7815	.7804	.7793	.7782	2	4	5	7	9
39	.7771	.7760	.7749	.7738	.7727	.7716	.7705	.7694	.7683	.7672	2	4	6	7	9
40	.7660	.7649	.7638	.7627	.7615	.7604	.7593	.7581	.7570	.7559	2	4	6	8	9
41	.7547	.7536	.7524	.7513	.7501	.7490	.7478	.7466	.7455	.7443	2	4	6	8	10
42	.7431	7420	.7408	.7396	.7385	.7373	.7361	.7349	.7337	.7325	2	4	6	8	10
43	.7314	.7302	.7290	.7278	.7266	.7254	.7242	.7230	.7218	.7206	2	4	6	8	10
44	.7193	.7181	.7169	.7157	.7145	.7133	.7120	.7108	.7096	.7083	2	4	6	8	10

The precise value of all cosines except cos 0° is less than 1.

NATURAL COSINES. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	— Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	.7071	.7059	.7046	.7034	.7022	.7009	.6997	.6984	.6972	.6959	2	4	6	8	10
46	.6947	.6934	.6921	.6909	.6896	.6884	.6871	.6858	.6845	.6833	2	4	6	8	11
47	.6820	.6807	.6794	.6782	.6769	.6756	.6743	.6730	.6717	.6704	2	4	6	9	11
48	.6691	.6678	.6665	.6652	.6639	.6626	.6613	.6600	.6587	.6574	2	4	7	9	11
49	.6561	.6547	.6534	.6521	.6508	.6494	.6481	.6468	.6455	.6441	2	4	7	9	11
50	.6428	.6414	.6401	.6388	.6374	.6361	.6347	.6334	.6320	.6307	2	4	7	9	11
51	.6293	.6280	.6266	.6252	.6239	.6225	.6211	.6198	.6184	.6170	2	5	7	9	11
52	.6157	.6143	.6129	.6115	.6101	.6088	.6074	.6060	.6046	.6032	2	5	7	9	12
53	.6018	.6004	.5990	.5976	.5962	.5948	.5934	.5920	.5906	.5892	2	5	7	9	12
54	.5878	.5864	.5850	.5835	.5821	.5807	.5793	.5779	.5764	.5750	2	5	7	9	12
55	.5736	.5721	.5707	.5693	.5678	.5664	.5650	.5635	.5621	.5606	2	5	7	10	12
56	.5592	.5577	.5563	.5548	.5534	.5519	.5505	.5490	.5476	.5461	2	5	7	10	12
57	.5446	.5432	.5417	.5402	.5388	.5373	.5358	.5344	.5329	.5314	2	5	7	10	12
58	.5299	.5284	.5270	.5255	.5240	.5225	.5210	.5195	.5180	.5165	2	5	7	10	12
59	.5150	.5135	.5120	.5105	.5090	.5075	.5060	.5045	.5030	.5015	3	5	8	10	13
60	.5000	.4985	.4970	.4955	.4939	.4924	.4909	.4894	.4879	.4863	3	5	8	10	13
61	.4848	.4833	.4818	.4802	.4787	.4772	.4756	.4741	.4726	.4710	3	5	8	10	13
62	.4695	.4679	.4664	.4648	.4633	.4617	.4602	.4586	.4571	.4555	3	5	8	10	13
63	.4540	.4524	.4509	.4493	.4478	.4462	.4446	.4431	.4415	.4399	3	5	8	10	13
64	.4384	.4368	.4352	.4337	.4321	.4305	.4289	.4274	.4258	.4242	3	5	8	11	13
65	.4226	.4210	.4195	.4179	.4163	.4147	.4131	.4115	.4099	.4083	3	5	8	11	13
66	.4067	.4051	.4035	.4019	.4003	.3987	.3971	.3955	.3939	.3923	3	5	8	11	13
67	.3907	.3891	.3875	.3859	.3843	.3827	.3811	.3795	.3778	.3762	3	5	8	11	13
68	.3746	.3730	.3714	.3697	.3681	.3665	.3649	.3633	.3616	.3600	3	5	8	11	14
69	.3584	.3567	.3551	.3535	.3518	.3502	.3486	.3469	.3453	.3437	3	5	8	11	14
70	.3420	.3404	.3387	.3371	.3355	.3338	.3322	.3305	.3289	.3272	3	5	8	11	14
71	.3256	.3239	.3223	.3206	.3190	.3173	.3156	.3140	.3123	.3107	3	6	8	11	14
72	.3090	.3074	.3057	.3040	.3024	.3007	.2990	.2974	.2957	.2940	3	6	8	11	14
73	.2924	.2907	.2890	.2874	.2857	.2840	.2823	.2807	.2790	.2773	3	6	8	11	14
74	.2756	.2740	.2723	.2706	.2689	.2672	.2656	.2639	.2622	.2605	3	6	8	11	14
75	.2588	.2571	.2554	.2538	.2521	.2504	.2487	.2470	.2453	.2436	3	6	8	11	14
76	.2419	.2402	.2385	.2368	.2351	.2334	.2317	.2300	.2284	.2267	3	6	8	11	14
77	.2250	.2233	.2215	.2198	.2181	.2164	.2147	.2130	.2113	.2096	3	6	9	11	14
78	.2079	.2062	.2045	.2028	.2011	.1994	.1977	.1959	.1942	.1925	3	6	9	11	14
79	.1908	.1891	.1874	.1857	.1840	.1822	.1805	.1788	.1771	.1754	3	6	9	11	14
80	.1736	.1719	.1702	.1685	.1668	.1650	.1633	.1616	.1599	.1582	3	6	9	11	14
81	.1564	.1547	.1530	.1513	.1495	.1478	.1461	.1444	.1426	.1409	3	6	9	12	14
82	.1392	.1374	.1357	.1340	.1323	.1305	.1288	.1271	.1253	.1236	3	6	9	12	14
83	.1219	.1201	.1184	.1167	.1149	.1132	.1115	.1097	.1080	.1063	3	6	9	12	14
84	.1045	.1028	.1011	.0993	.0976	.0958	.0941	.0924	.0906	.0889	3	6	9	12	14
85	.0872	.0854	.0837	.0819	.0802	.0785	.0767	.0750	.0732	.0715	3	6	9	12	14
86	.0698	.0680	.0663	.0645	.0628	.0610	.0593	.0576	.0558	.0541	3	6	9	12	15
87	.0523	.0506	.0488	.0471	.0454	.0436	.0419	.0401	.0384	.0366	3	6	9	12	15
88	.0349	.0332	.0314	.0297	.0279	.0262	.0244	.0227	.0209	.0192	3	6	9	12	15
89	.0175	.0157	.0140	.0122	.0105	.0087	.0070	.0052	.0035	.0017	3	6	9	12	15

All the above cosines are less than 1.

NATURAL TANGENTS. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	0.0000	.0017	.0035	.0052	.0070	.0087	.0105	.0122	.0140	.0157	3	6	9	12	15
1	0.0175	.0192	.0209	.0227	.0244	.0262	.0279	.0297	.0314	.0332	3	6	9	12	15
2	0.0349	.0367	.0384	.0402	.0419	.0437	.0454	.0472	.0489	.0507	3	6	9	12	15
3	0.0524	.0542	.0559	.0577	.0594	.0612	.0629	.0647	.0664	.0682	3	6	9	12	15
4	0.0699	.0717	.0734	.0752	.0769	.0787	.0805	.0822	.0840	.0857	3	6	9	12	15
5	0.0875	.0892	.0910	.0928	.0945	.0963	.0981	.0998	.1016	.1033	3	6	9	12	15
6	0.1051	.1069	.1086	.1104	.1122	.1139	.1157	.1175	.1192	.1210	3	6	9	12	15
7	0.1228	.1246	.1263	.1281	.1299	.1317	.1334	.1352	.1370	.1388	3	6	9	12	15
8	0.1405	.1423	.1441	.1459	.1477	.1495	.1512	.1530	.1548	.1566	3	6	9	12	15
9	0.1584	.1602	.1620	.1638	.1655	.1673	.1691	.1709	.1727	.1745	3	6	9	12	15
10	0.1763	.1781	.1799	.1817	.1835	.1853	.1871	.1890	.1908	.1926	3	6	9	12	15
11	0.1944	.1962	.1980	.1998	.2016	.2035	.2053	.2071	.2089	.2107	3	6	9	12	15
12	0.2126	.2144	.2162	.2180	.2199	.2217	.2235	.2254	.2272	.2290	3	6	9	12	15
13	0.2309	.2327	.2345	.2364	.2382	.2401	.2419	.2438	.2456	.2475	3	6	9	12	15
14	0.2493	.2512	.2530	.2549	.2568	.2586	.2605	.2623	.2642	.2661	3	6	9	12	16
15	0.2679	.2698	.2717	.2736	.2754	.2773	.2792	.2811	.2830	.2849	3	6	9	13	16
16	0.2867	.2886	.2905	.2924	.2943	.2962	.2981	.3000	.3019	.3038	3	6	9	13	16
17	0.3057	.3076	.3096	.3115	.3134	.3153	.3172	.3191	.3211	.3230	3	6	10	13	16
18	0.3249	.3269	.3288	.3307	.3327	.3346	.3365	.3385	.3404	.3424	3	6	10	13	16
19	0.3443	.3463	.3482	.3502	.3522	.3541	.3561	.3581	.3600	.3620	3	7	10	13	16
20	0.3640	.3659	.3679	.3699	.3719	.3739	.3759	.3779	.3799	.3819	3	7	10	13	17
21	0.3839	.3859	.3879	.3899	.3919	.3939	.3959	.3979	.4000	.4020	3	7	10	13	17
22	0.4040	.4061	.4081	.4101	.4122	.4142	.4163	.4183	.4204	.4224	3	7	10	14	17
23	0.4245	.4265	.4286	.4307	.4327	.4348	.4369	.4390	.4411	.4431	3	7	10	14	17
24	0.4452	.4473	.4494	.4515	.4536	.4557	.4578	.4599	.4621	.4642	4	7	11	14	18
25	0.4663	.4684	.4706	.4727	.4748	.4770	.4791	.4813	.4834	.4856	4	7	11	14	18
26	0.4877	.4899	.4921	.4942	.4964	.4986	.5008	.5029	.5051	.5073	4	7	11	15	18
27	0.5095	.5117	.5139	.5161	.5184	.5206	.5228	.5250	.5272	.5295	4	7	11	15	18
28	0.5317	.5340	.5362	.5384	.5407	.5430	.5452	.5475	.5498	.5520	4	8	11	15	19
29	0.5543	.5566	.5589	.5612	.5635	.5658	.5681	.5704	.5727	.5750	4	8	12	15	19
30	0.5774	.5797	.5820	.5844	.5867	.5890	.5914	.5938	.5961	.5985	4	8	12	16	20
31	0.6009	.6032	.6056	.6080	.6104	.6128	.6152	.6176	.6200	.6224	4	8	12	16	20
32	0.6249	.6273	.6297	.6322	.6346	.6371	.6395	.6420	.6445	.6469	4	8	12	16	20
33	0.6494	.6519	.6544	.6569	.6594	.6619	.6644	.6669	.6694	.6720	4	8	13	17	21
34	0.6745	.6771	.6796	.6822	.6847	.6873	.6899	.6924	.6950	.6976	4	9	13	17	21
35	0.7002	.7028	.7054	.7080	.7107	.7133	.7159	.7186	.7212	.7239	4	9	13	18	22
36	0.7265	.7292	.7319	.7346	.7373	.7400	.7427	.7454	.7481	.7508	5	9	14	18	23
37	0.7536	.7563	.7590	.7618	.7646	.7673	.7701	.7729	.7757	.7785	5	9	14	18	23
38	0.7813	.7841	.7869	.7898	.7926	.7954	.7983	.8012	.8040	.8069	5	9	14	19	24
39	0.8098	.8127	.8156	.8185	.8214	.8243	.8273	.8302	.8332	.8361	5	10	15	20	24
40	0.8391	.8421	.8451	.8481	.8511	.8541	.8571	.8601	.8632	.8662	5	10	15	20	25
41	0.8693	.8724	.8754	.8785	.8816	.8847	.8878	.8910	.8941	.8972	5	10	16	21	26
42	0.9004	.9036	.9067	.9099	.9131	.9163	.9195	.9228	.9260	.9293	5	11	16	21	27
43	0.9325	.9358	.9391	.9424	.9457	.9490	.9523	.9556	.9590	.9623	6	11	17	22	28
44	0.9657	.9691	.9725	.9759	.9793	.9827	.9861	.9896	.9930	.9965	6	11	17	23	29

All tangents less than tan 45° are less than 1.

NATURAL TANGENTS. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	1.0000	.0035	.0070	.0105	.0141	.0176	.0212	.0247	.0283	.0319	6	12	18	24	30
46	1.0355	.0392	.0428	.0464	.0501	.0538	.0575	.0612	.0649	.0686	6	12	18	25	31
47	1.0724	.0761	.0799	.0837	.0875	.0913	.0951	.0990	.1028	.1067	6	13	19	25	32
48	1.1106	.1145	.1184	.1224	.1263	.1303	.1343	.1383	.1423	.1463	7	13	20	26	33
49	1.1504	.1544	.1585	.1626	.1667	.1708	.1750	.1792	.1833	.1875	7	14	21	28	34
50	1.1918	.1960	.2002	.2045	.2088	.2131	.2174	.2218	.2261	.2305	7	14	22	29	36
51	1.2349	.2393	.2437	.2482	.2527	.2572	.2617	.2662	.2708	.2753	8	15	23	30	38
52	1.2799	.2846	.2892	.2938	.2985	.3032	.3079	.3127	.3175	.3222	8	16	24	31	39
53	1.3270	.3319	.3367	.3416	.3465	.3514	.3564	.3613	.3663	.3713	8	16	25	33	41
54	1.3764	.3814	.3865	.3916	.3968	.4019	.4071	.4124	.4176	.4229	9	17	26	34	43
55	1.4281	.4335	.4388	.4442	.4496	.4550	.4605	.4659	.4715	.4770	9	18	27	36	45
56	1.4826	.4882	.4938	.4994	.5051	.5108	.5166	.5224	.5282	.5340	10	19	29	38	48
57	1.5399	.5458	.5517	.5577	.5637	.5697	.5757	.5818	.5880	.5941	10	20	30	40	50
58	1.6003	.6066	.6128	.6191	.6255	.6319	.6383	.6447	.6512	.6577	11	21	32	43	53
59	1.6643	.6709	.6775	.6842	.6909	.6977	.7045	.7113	.7182	.7251	11	23	34	45	56
60	1.7321	.7391	.7461	.7532	.7603	.7675	.7747	.7820	.7893	.7966	12	24	36	48	60
61	1.8040	.8115	.8190	.8265	.8341	.8418	.8495	.8572	.8650	.8728	13	26	38	51	64
62	1.8807	.8887	.8967	.9047	.9128	.9210	.9292	.9375	.9458	.9542	14	27	41	55	68
63	1.9626	.9711	.9797	.9883	.9970	**.0057**	.0145	.0233	.0323	.0413	15	29	44	58	73
64	2.0503	.0594	.0686	.0778	.0872	.0965	.1060	.1155	.1251	.1348	16	31	47	63	78
65	2.1445	.1543	.1642	.1742	.1842	.1943	.2045	.2148	.2251	.2355	17	34	51	68	85
66	2.2460	.2566	.2673	.2781	.2889	.2998	.3109	.3220	.3332	.3445	18	37	55	73	92
67	2.3559	.3673	.3789	.3906	.4023	.4142	.4262	.4383	.4504	.4627	20	40	60	79	99
68	2.4751	.4876	.5002	.5129	.5257	.5386	.5517	.5649	.5782	.5916	22	43	65	87	108
69	2.6051	.6187	.6325	.6464	.6605	.6746	.6889	.7034	.7179	.7326	24	47	71	95	119
70	2.7475	.7625	.7776	.7929	.8083	.8239	.8397	.8556	.8716	.8878	26	52	78	104	130
71	2.9042	.9208	.9375	.9544	.9714	.9887	**.0061**	.0237	.0415	.0595	29	58	87	116	144
72	3.0777	.0961	.1146	.1334	.1524	.1716	.1910	.2106	.2305	.2506	32	64	96	129	161
73	3.2709	.2914	.3122	.3332	.3544	.3759	.3977	.4197	.4420	.4646	36	72	108	144	180
74	3.4874	.5105	.5339	.5576	.5816	.6059	.6305	.6554	.6806	.7062	41	81	122	163	204
75	3.7321	.7583	.7848	.8118	.8391	.8667	.8947	.9232	.9520	.9812					
76	4.0108	.0408	.0713	.1022	.1335	.1653	.1976	.2303	.2635	.2972					
77	4.3315	.3662	.4015	.4373	.4737	.5107	.5483	.5864	.6252	.6646					
78	4.7046	.7453	.7867	.8288	.8716	.9152	.9594	**.0045**	.0504	.0970	Use ordinary interpolation.				
79	5.1446	.1929	.2422	.2924	.3435	.3955	.4486	.5026	.5578	.6140					
80	5.6713	.7297	.7894	.8502	.9124	.9758	**.0405**	.1066	.1742	.2432					
81	6.3138	.3859	.4596	.5350	.6122	.6912	.7720	.8548	.9395	**.0264**					
82	7.1154	.2066	.3002	.3962	.4947	.5958	.6996	.8062	.9158	**.0285**					
83	8.1443	.2636	.3863	.5126	.6427	.7769	.9152	**.0579**	.2052	.3572					
84	9.5144	.6768	.8448	**.0187**	.1988	.3854	.5789	.7797	.9882	**.2048**					
85	11.430	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95					
86	14.301	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46					
87	19.081	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27					
88	28.636	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08					
89	57.290	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0					

The integral part of tangents in heavy-face type is 1 greater than the preceding part.

NATURAL COTANGENTS. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	— Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	∞	573.0	286.5	191.0	143.2	114.6	95.49	81.85	71.62	63.66					
1	57.290	52.08	47.74	44.07	40.92	38.19	35.80	33.69	31.82	30.14					
2	28.636	27.27	26.03	24.90	23.86	22.90	22.02	21.20	20.45	19.74					
3	19.081	18.46	17.89	17.34	16.83	16.35	15.89	15.46	15.06	14.67					
4	14.301	13.95	13.62	13.30	13.00	12.71	12.43	12.16	11.91	11.66	Use ordinary interpolation.				
5	11.430	.2048	**.9882**	.7797	.5789	.3854	.1988	.0187	**.8448**	.6768					
6	9.5144	.3572	.2052	.0579	**.9152**	.7769	.6427	.5126	.3863	.2636					
7	8.1443	.0285	**.9158**	.8062	.6996	.5958	.4947	.3962	.3002	.2066					
8	7.1154	.0264	**.9395**	.8548	.7720	.6912	.6122	.5350	.4596	.3859					
9	6.3138	.2432	.1742	.1066	.0405	**.9758**	.9124	.8502	.7894	.7297					
10	5.6713	.6140	.5578	.5026	.4486	.3955	.3435	.2924	.2422	.1929					
11	5.1446	.0970	.0504	.0045	**.9594**	.9152	.8716	.8288	.7867	.7453					
12	4.7046	.6646	.6252	.5864	.5483	.5107	.4737	.4373	.4015	.3662					
13	4.3315	.2972	.2635	.2303	.1976	.1653	.1335	.1022	.0713	.0408					
14	4.0108	**.9812**	.9520	.9232	.8947	.8667	.8391	.8118	.7848	.7583					
15	3.7321	.7062	.6806	.6554	.6305	.6059	.5816	.5576	.5339	.5105	41	81	122	163	203
16	3.4874	.4646	.4420	.4197	.3977	.3759	.3544	.3332	.3122	.2914	36	72	108	144	180
17	3.2709	.2506	.2305	.2106	.1910	.1716	.1524	.1334	.1146	.0961	32	64	96	129	161
18	3.0777	.0595	.0415	.0237	.0061	**.9887**	.9714	.9544	.9375	.9208	29	58	87	116	144
19	2.9042	.8878	.8716	.8556	.8397	.8239	.8083	.7929	.7776	.7625	26	52	78	104	130
20	2.7475	.7326	.7179	.7034	.6889	.6746	.6605	.6464	.6325	.6187	24	47	71	95	119
21	2.6051	.5916	.5782	.5649	.5517	.5386	.5257	.5129	.5002	.4876	22	43	65	87	108
22	2.4751	.4627	.4504	.4383	.4262	.4142	.4023	.3906	.3789	.3673	20	40	60	79	99
23	2.3559	.3445	.3332	.3220	.3109	.2998	.2889	.2781	.2673	.2566	18	37	55	73	92
24	2.2460	.2355	.2251	.2148	.2045	.1943	.1842	.1742	.1642	.1543	17	34	51	68	85
25	2.1445	.1348	.1251	.1155	.1060	.0965	.0872	.0778	.0686	.0594	16	31	47	63	78
26	2.0503	.0413	.0323	.0233	.0145	.0057	**.9970**	.9883	9797	.9711	15	29	44	58	73
27	1.9626	.9542	.9458	.9375	.9292	.9210	.9128	.9047	.8967	.8887	14	27	41	55	68
28	1.8807	.8728	.8650	.8572	.8495	.8418	.8341	.8265	.8190	.8115	13	26	38	51	64
29	1.8040	.7966	.7893	.7820	.7747	.7675	.7603	.7532	.7461	.7391	12	24	36	48	60
30	1.7321	.7251	.7182	.7113	.7045	.6977	.6909	.6842	.6775	.6709	11	23	34	45	56
31	1.6643	.6577	.6512	.6447	.6383	.6319	.6255	.6191	.6128	.6066	11	21	32	43	53
32	1.6003	.5941	.5880	.5818	.5757	.5697	.5637	.5577	.5517	.5458	10	20	30	40	50
33	1.5399	.5340	.5282	.5224	.5166	.5108	.5051	.4994	.4938	.4882	10	19	29	38	48
34	1.4826	.4770	.4715	.4659	.4605	.4550	.4496	.4442	.4388	.4335	9	18	27	36	45
35	1.4281	.4229	.4176	.4124	.4071	.4019	.3968	.3916	.3865	.3814	9	17	26	34	43
36	1.3764	.3713	.3663	.3613	.3564	.3514	.3465	.3416	.3367	.3319	8	16	25	33	41
37	1.3270	.3222	.3175	.3127	.3079	.3032	.2985	.2938	.2892	.2846	8	16	24	31	39
38	1.2799	.2753	.2708	.2662	.2617	.2572	.2527	.2482	.2437	.2393	8	15	23	30	38
39	1.2349	.2305	.2261	.2218	.2174	.2131	.2088	.2045	.2002	.1960	7	14	22	29	36
40	1.1918	.1875	.1833	.1792	.1750	.1708	.1667	.1626	.1585	.1544	7	14	21	28	34
41	1.1504	.1463	.1423	.1383	.1343	.1303	.1263	.1224	.1184	.1145	7	13	20	26	33
42	1.1106	.1067	.1028	.0990	.0951	.0913	.0875	.0837	.0799	.0761	6	13	19	25	32
43	1.0724	.0686	.0649	.0612	.0575	.0538	.0501	.0464	.0428	.0392	6	12	18	25	31
44	1.0355	.0319	.0283	.0247	.0212	.0176	.0141	.0105	.0070	.0035	6	12	18	24	30

Heavy-face type indicates that the integral part is to be decreased by 1.

NATURAL COTANGENTS. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	— Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	1.0000	.9965	.9930	.9896	.9861	.9827	.9793	.9759	.9725	.9691	6	11	17	23	29
46	0.9657	.9623	.9590	.9556	.9523	.9490	.9457	.9424	.9391	.9358	6	11	17	22	28
47	0.9325	.9293	.9260	.9228	.9195	.9163	.9131	.9099	.9067	.9036	5	11	16	21	27
48	0.9004	.8972	.8941	.8910	.8878	.8847	.8816	.8785	.8754	.8724	5	10	16	21	26
49	0.8693	.8662	.8632	.8601	.8571	.8541	.8511	.8481	.8451	.8421	5	10	15	20	25
50	0.8391	.8361	.8332	.8302	.8273	.8243	.8214	.8185	.8156	.8127	5	10	15	20	24
51	0.8098	.8069	.8040	.8012	.7983	.7954	.7926	.7898	.7869	.7841	5	9	14	19	24
52	0.7813	.7785	.7757	.7729	.7701	.7673	.7646	.7618	.7590	.7563	5	9	14	18	23
53	0.7536	.7508	.7481	.7454	.7427	.7400	.7373	.7346	.7319	.7292	5	9	14	18	23
54	0.7265	.7239	.7212	.7186	.7159	.7133	.7107	.7080	.7054	.7028	4	9	13	18	22
55	0.7002	.6976	.6950	.6924	.6899	.6873	.6847	.6822	.6796	.6771	4	9	13	17	21
56	0.6745	.6720	.6694	.6669	.6644	.6619	.6594	.6569	.6544	.6519	4	8	13	17	21
57	0.6494	.6469	.6445	.6420	.6395	.6371	.6346	.6322	.6297	.6273	4	8	12	16	20
58	0.6249	.6224	.6200	.6176	.6152	.6128	.6104	.6080	.6056	.6032	4	8	12	16	20
59	0.6009	.5985	.5961	.5938	.5914	.5890	.5867	.5844	.5820	.5797	4	8	12	16	20
60	0.5774	.5750	.5727	.5704	.5681	.5658	.5635	.5612	.5589	.5566	4	8	12	15	19
61	0.5543	.5520	.5498	.5475	.5452	.5430	.5407	.5384	.5362	.5340	4	8	11	15	19
62	0.5317	.5295	.5272	.5250	.5228	.5206	.5184	.5161	.5139	.5117	4	7	11	15	18
63	0.5095	.5073	.5051	.5029	.5008	.4986	.4964	.4942	.4921	.4899	4	7	11	15	18
64	0.4877	.4856	.4834	.4813	.4791	.4770	.4748	.4727	.4706	.4684	4	7	11	14	18
65	0.4663	.4642	.4621	.4599	.4578	.4557	.4536	.4515	.4494	.4473	4	7	11	14	18
66	0.4452	.4431	.4411	.4390	.4369	.4348	.4327	.4307	.4286	.4265	3	7	10	14	17
67	0.4245	.4224	.4204	.4183	.4163	.4142	.4122	.4101	.4081	.4061	3	7	10	14	17
68	0.4040	.4020	.4000	.3979	.3959	.3939	.3919	.3899	.3879	.3859	3	7	10	13	17
69	0.3839	.3819	.3799	.3779	.3759	.3739	.3719	.3699	.3679	.3659	3	7	10	13	17
70	0.3640	.3620	.3600	.3581	.3561	.3541	.3522	.3502	.3482	.3463	3	7	10	13	16
71	0.3443	.3424	.3404	.3385	.3365	.3346	.3327	.3307	.3288	.3269	3	6	10	13	16
72	0.3249	.3230	.3211	.3191	.3172	.3153	.3134	.3115	.3096	.3076	3	6	10	13	16
73	0.3057	.3038	.3019	.3000	.2981	.2962	.2943	.2924	.2905	.2886	3	6	9	13	16
74	0.2867	.2849	.2830	.2811	.2792	.2773	.2754	.2736	.2717	.2698	3	6	9	13	16
75	0.2679	.2661	.2642	.2623	.2605	.2586	.2568	.2549	.2530	.2512	3	6	9	12	16
76	0.2493	.2475	.2456	.2438	.2419	.2401	.2382	.2364	.2345	.2327	3	6	9	12	15
77	0.2309	.2290	.2272	.2254	.2235	.2217	.2199	.2180	.2162	.2144	3	6	9	12	15
78	0.2126	.2107	.2089	.2071	.2053	.2035	.2016	.1998	.1980	.1962	3	6	9	12	15
79	0.1944	.1926	.1908	.1890	.1871	.1853	.1835	.1817	.1799	.1781	3	6	9	12	15
80	0.1763	.1745	.1727	.1709	.1691	.1673	.1655	.1638	.1620	.1602	3	6	9	12	15
81	0.1584	.1566	.1548	.1530	.1512	.1495	.1477	.1459	.1441	.1423	3	6	9	12	15
82	0.1405	.1388	.1370	.1352	.1334	.1317	.1299	.1281	.1263	.1246	3	6	9	12	15
83	0.1228	.1210	.1192	.1175	.1157	.1139	.1122	.1104	.1086	.1069	3	6	9	12	15
84	0.1051	.1033	.1016	.0998	.0981	.0963	.0945	.0928	.0910	.0892	3	6	9	12	15
85	0.0875	.0857	.0840	.0822	.0805	.0787	.0769	.0752	.0734	.0717	3	6	9	12	15
86	0.0699	.0682	.0664	.0647	.0629	.0612	.0594	.0577	.0559	.0542	3	6	9	12	15
87	0.0524	.0507	.0489	.0472	.0454	.0437	.0419	.0402	.0384	.0367	3	6	9	12	15
88	0.0349	.0332	.0314	.0297	.0279	.0262	.0244	.0227	.0209	.0192	3	6	9	12	15
89	0.0175	.0157	.0140	.0122	.0105	.0087	.0070	.0052	.0035	.0017	3	6	9	12	15

All cotangents greater than cot 45° are less than 1.

NATURAL SECANTS. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	1.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	0	0	0	0	0
1	1.0002	.0002	.0002	.0003	.0003	.0003	.0004	.0004	.0005	.0006	0	0	0	0	0
2	1.0006	.0007	.0007	.0008	.0009	.0010	.0010	.0011	.0012	.0013	0	0	0	0	1
3	1.0014	.0015	.0016	.0017	.0018	.0019	.0020	.0021	.0022	.0023	0	0	1	1	1
4	1.0024	.0026	.0027	.0028	.0030	.0031	.0032	.0034	.0035	.0037	0	0	1	1	1
5	1.0038	.0040	.0041	.0043	.0045	.0046	.0048	.0050	.0051	.0053	0	1	1	1	1
6	1.0055	.0057	.0059	.0061	.0063	.0065	.0067	.0069	.0071	.0073	0	1	1	1	2
7	1.0075	.0077	.0079	.0082	.0084	.0086	.0089	.0091	.0093	.0096	0	1	1	2	2
8	1.0098	.0101	.0103	.0106	.0108	.0111	.0114	.0116	.0119	.0122	0	1	1	2	2
9	1.0125	.0127	.0130	.0133	.0136	.0139	.0142	.0145	.0148	.0151	0	1	1	2	2
10	1.0154	.0157	.0161	.0164	.0167	.0170	.0174	.0177	.0180	.0184	1	1	2	2	3
11	1.0187	.0191	.0194	.0198	.0201	.0205	.0209	.0212	.0216	.0220	1	1	2	2	3
12	1.0223	.0227	.0231	.0235	.0239	.0243	.0247	.0251	.0255	.0259	1	1	2	3	3
13	1.0263	.0267	.0271	.0276	.0280	.0284	.0288	.0293	.0297	.0302	1	1	2	3	4
14	1.0306	.0311	.0315	.0320	.0324	.0329	.0334	.0338	.0343	.0348	1	2	2	3	4
15	1.0353	.0358	.0363	.0367	.0372	.0377	.0382	.0388	.0393	.0398	1	2	3	3	4
16	1.0403	.0408	.0413	.0419	.0424	.0429	.0435	.0440	.0446	.0451	1	2	3	4	4
17	1.0457	.0463	.0468	.0474	.0480	.0485	.0491	.0497	.0503	.0509	1	2	3	4	5
18	1.0515	.0521	.0527	.0533	.0539	.0545	.0551	.0557	.0564	.0570	1	2	3	4	5
19	1.0576	.0583	.0589	.0595	.0602	.0608	.0615	.0622	.0628	.0635	1	2	3	4	5
20	1.0642	.0649	.0655	.0662	.0669	.0676	.0683	.0690	.0697	.0704	1	2	3	5	6
21	1.0711	.0719	.0726	.0733	.0740	.0748	.0755	.0763	.0770	.0778	1	3	4	5	6
22	1.0785	.0793	.0801	.0808	.0816	.0824	.0832	.0840	.0848	.0856	1	3	4	5	7
23	1.0864	.0872	.0880	.0888	.0896	.0904	.0913	.0921	.0929	.0938	1	3	4	6	7
24	1.0946	.0955	.0963	.0972	.0981	.0989	.0998	.1007	.1016	.1025	1	3	4	6	7
25	1.1034	.1043	.1052	.1061	.1070	.1079	.1089	.1098	.1107	.1117	2	3	5	6	8
26	1.1126	.1136	.1145	.1155	.1164	.1174	.1184	.1194	.1203	.1213	2	3	5	6	8
27	1.1223	.1233	.1243	.1253	.1264	.1274	.1284	.1294	.1305	.1315	2	3	5	7	9
28	1.1326	.1336	.1347	.1357	.1368	.1379	.1390	.1401	.1412	.1423	2	4	5	7	9
29	1.1434	.1445	.1456	.1467	.1478	.1490	.1501	.1512	.1524	.1535	2	4	6	8	9
30	1.1547	.1559	.1570	.1582	.1594	.1606	.1618	.1630	.1642	.1654	2	4	6	8	10
31	1.1666	.1679	.1691	.1703	.1716	.1728	.1741	.1753	.1765	.1779	2	4	6	8	10
32	1.1792	.1805	.1818	.1831	.1844	.1857	.1870	.1883	.1897	.1910	2	4	7	9	11
33	1.1924	.1937	.1951	.1964	.1978	.1992	.2006	.2020	.2034	.2048	2	5	7	9	12
34	1.2062	.2076	.2091	.2105	.2120	.2134	.2149	.2163	.2178	.2193	2	5	7	10	12
35	1.2208	.2223	.2238	.2253	.2268	.2283	.2299	.2314	.2329	.2345	3	5	8	10	13
36	1.2361	.2376	.2392	.2408	.2424	.2440	.2456	.2472	.2489	.2505	3	5	8	11	13
37	1.2521	.2538	.2554	.2571	.2588	.2605	.2622	.2639	.2656	.2673	3	6	8	11	14
38	1.2690	.2708	.2725	.2742	.2760	.2778	.2796	.2813	.2831	.2849	3	6	9	12	15
39	1.2868	.2886	.2904	.2923	.2941	.2960	.2978	.2997	.3016	.3035	3	6	9	12	16
40	1.3054	.3073	.3093	.3112	.3131	.3151	.3171	.3190	.3210	.3230	3	7	10	13	16
41	1.3250	.3270	.3291	.3311	.3331	.3352	.3373	.3393	.3414	.3435	3	7	10	14	17
42	1.3456	.3478	.3499	.3520	.3542	.3563	.3585	.3607	.3629	.3651	4	7	11	14	18
43	1.3673	.3696	.3718	.3741	.3763	.3786	.3809	.3832	.3855	.3878	4	8	11	15	19
44	1.3902	.3925	.3949	.3972	.3996	.4020	.4044	.4069	.4093	.4118	4	8	12	16	20

The integral part of each of the above secants is 1.

NATURAL SECANTS. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	1.4142	.4167	.4192	.4217	.4242	.4267	.4293	.4318	.4344	.4370	4	8	13	17	21
46	1.4396	.4422	.4448	.4474	.4501	.4527	.4554	.4581	.4608	.4635	4	9	13	18	22
47	1.4663	.4690	.4718	.4746	.4774	.4802	.4830	.4859	.4887	.4916	5	9	14	19	23
48	1.4945	.4974	.5003	.5032	.5062	.5092	.5121	.5151	.5182	.5212	5	10	15	20	25
49	1.5243	.5273	.5304	.5335	.5366	.5398	.5429	.5461	.5493	.5525	5	10	16	21	26
50	1.5557	.5590	.5622	.5655	.5688	.5721	.5755	.5788	.5822	.5856	6	11	17	22	28
51	1.5890	.5925	.5959	.5994	.6029	.6064	.6099	.6135	.6171	.6207	6	12	18	24	29
52	1.6243	.6279	.6316	.6353	.6390	.6427	.6464	.6502	.6540	.6578	6	12	19	25	31
53	1.6616	.6655	.6694	.6733	.6772	.6812	.6852	.6892	.6932	.6972	7	13	20	26	33
54	1.7013	.7054	.7095	.7137	.7179	.7221	.7263	.7305	.7348	.7391	7	14	21	28	35
55	1.7434	.7478	.7522	.7566	.7610	.7655	.7700	.7745	.7791	.7837	7	15	22	30	37
56	1.7883	.7929	.7976	.8023	.8070	.8118	.8166	.8214	.8263	.8312	8	16	24	32	40
57	1.8361	.8410	.8460	.8510	.8561	.8612	.8663	.8714	.8766	.8818	8	17	26	34	43
58	1.8871	.8924	.8977	.9031	.9084	.9139	.9194	.9249	.9304	.9360	9	18	27	36	45
59	1.9416	.9473	.9530	.9587	.9645	.9703	.9762	.9821	.9880	.9940	10	19	29	39	49
60	2.0000	.0061	.0122	.0183	.0245	.0308	.0371	.0434	.0498	.0562	10	21	31	42	52
61	2.0627	.0692	.0757	.0824	.0890	.0957	.1025	.1093	.1162	.1231	11	22	34	45	56
62	2.1301	.1371	.1441	.1513	.1584	.1657	.1730	.1803	.1877	.1952	12	24	36	48	61
63	2.2027	.2103	.2179	.2256	.2333	.2412	.2490	.2570	.2650	.2730	13	26	39	52	65
64	2.2812	.2894	.2976	.3060	.3144	.3228	.3314	.3400	.3486	.3574	14	28	43	57	71
65	2.3662	.3751	.3841	.3931	.4022	.4114	.4207	.4300	.4395	.4490	15	31	46	62	77
66	2.4586	.4683	.4780	.4879	.4978	.5078	.5180	.5282	.5384	.5488	17	34	50	67	84
67	2.5593	.5699	.5805	.5913	.6022	.6131	.6242	.6354	.6466	.6580	18	37	55	73	92
68	2.6695	.6811	.6927	.7046	.7165	.7285	.7407	.7529	.7653	.7778	20	40	60	81	101
69	2.7904	.8032	.8161	.8291	.8422	.8555	.8688	.8824	.8960	.9099	22	45	67	89	111
70	2.9238	.9379	.9521	.9665	.9811	.9957	**.0106**	.0256	.0407	.0561	25	49	74	98	123
71	3.0716	.0872	.1030	.1190	.1352	.1515	.1681	.1848	.2017	.2188	27	55	82	110	137
72	3.2361	.2535	.2712	.2891	.3072	.3255	.3440	.3628	.3817	.4009	31	61	92	123	153
73	3.4203	.4399	.4598	.4799	.5003	.5209	.5418	.5629	.5843	.6060	35	69	104	138	173
74	3.6280	.6502	.6727	.6955	.7186	.7420	.7657	.7897	.8140	.8387	39	78	118	156	196
75	3.8637	.8890	.9147	.9408	.9672	.9939	**.0211**	.0486	.0765	.1048					
76	4.1336	.1627	.1923	.2223	.2527	.2837	.3150	.3469	.3792	.4121					
77	4.4454	.4793	.5137	.5486	.5841	.6202	.6569	.6942	.7321	.7706					
78	4.8097	.8496	.8901	.9313	.9732	**.0159**	.0593	.1034	.1484	.1942					
79	5.2408	.2883	.3367	.3860	.4362	.4874	.5396	.5928	.6470	.7023					
80	5.7588	.8164	.8751	.9351	.9963	**.0589**	.1227	.1880	.2546	.3228					
81	6.3925	.4637	.5366	.6111	.6874	.7655	.8454	.9273	**.0112**	.0972	Use ordinary interpolation.				
82	7.1853	.2757	.3684	.4635	.5611	.6613	.7642	.8700	.9787	**.0905**					
83	8.2055	.3238	.4457	.5711	.7004	.8337	.9711	**.1129**	.2593	.4105					
84	9.5668	.7283	.8955	**.0684**	.2479	.4336	.6261	.8260	**.0336**	.2492					
85	11.474	11.71	11.95	12.20	12.47	12.74	13.03	13.34	13.65	13.99					
86	14.336	14.70	15.09	15.50	15.93	16.38	16.86	17.37	17.91	18.49					
87	19.107	19.77	20.47	21.23	22.04	22.93	23.88	24.92	26.05	27.29					
88	28.654	30.16	31.84	33.71	35.81	38.20	40.93	44.08	47.75	52.09					
89	57.299	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0					

Heavy-face type indicates that the integral part is to be increased by 1.

NATURAL COSECANTS. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	– Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	∞	573.0	286.5	191.0	143.2	114.6	95.49	81.85	71.62	63.66					
1	57.299	52.09	47.75	44.08	40.93	38.20	35.81	33.71	31.84	30.16					
2	28.654	27.29	26.05	24.92	23.88	22.93	22.04	21.23	20.47	19.77					
3	19.107	18.49	17.91	17.37	16.86	16.38	15.93	15.50	15.09	14.70					
4	14.336	13.99	13.65	13.34	13.03	12.74	12.47	12.20	11.95	11.71					
5	11.474	.2492	.0336	**.8260**	.6261	.4336	.2479	.0684	**.8955**	.7283					
6	9.5668	.4105	.2593	.1129	**.9711**	.8337	.7004	.5711	.4457	.3238					
7	8.2055	.0905	**.9787**	.8700	.7042	.6613	.5611	.4635	.3684	.2757	Use ordinary interpolation.				
8	7.1853	.0972	.0112	**.9273**	.8454	.7655	.6874	.6111	.5366	.4637					
9	6.3925	.3228	.2546	.1880	.1227	.0589	**.9963**	.9351	.8751	.8164					
10	5.7588	.7023	.6470	.5928	.5396	.4874	.4362	.3860	.3367	.2883					
11	5.2408	.1942	.1484	.1034	.0593	.0159	**.9732**	.9313	.8901	.8496					
12	4.8097	.7706	.7321	.6942	.6569	.6202	.5841	.5486	.5137	.4793					
13	4.4454	.4121	.3792	.3469	.3150	.2837	.2527	.2223	.1923	.1627					
14	4.1336	.1048	.0765	.0486	.0211	**.9939**	.9672	.9408	.9147	.8890					
15	3.8637	.8387	.8140	.7897	.7657	.7420	.7186	.6955	.6727	.6502	39	79	118	157	196
16	3.6280	.6060	.5843	.5629	.5418	.5209	.5003	.4799	.4598	.4399	35	69	104	138	173
17	3.4203	.4009	.3817	.3628	.3440	.3255	.3072	.2891	.2712	.2535	31	61	92	123	153
18	3.2361	.2188	.2017	.1848	.1681	.1515	.1352	.1190	.1030	.0872	27	55	82	110	137
19	3.0716	.0561	.0407	.0256	.0106	**.9957**	.9811	.9665	.9521	.9379	25	49	74	98	123
20	2.9238	.9099	.8960	.8824	.8688	.8555	.8422	.8291	.8161	.8032	22	45	67	89	111
21	2.7904	.7778	.7653	.7529	.7407	.7285	.7165	.7046	.6927	.6811	20	40	60	81	101
22	2.6695	.6580	.6466	.6354	.6242	.6131	.6022	.5913	.5805	.5699	18	37	55	73	92
23	2.5593	.5488	.5384	.5282	.5180	.5078	.4978	.4879	.4780	.4683	17	34	50	67	84
24	2.4586	.4490	.4395	.4300	.4207	.4114	.4022	.3931	.3841	.3751	15	31	46	62	77
25	2.3662	.3574	.3486	.3400	.3314	.3228	.3144	.3060	.2976	.2894	14	28	43	57	71
26	2.2812	.2730	.2650	.2570	.2490	.2412	.2333	.2256	.2179	.2103	13	26	39	52	65
27	2.2027	.1952	.1877	.1803	.1730	.1657	.1584	.1513	.1441	.1371	12	24	36	48	61
28	2.1301	.1231	.1162	.1093	.1025	.0957	.0890	.0824	.0757	.0692	11	22	34	45	56
29	2.0627	.0562	.0498	.0434	.0371	.0308	.0245	.0183	.0122	.0061	10	21	31	42	52
30	2.0000	**.9940**	.9880	.9821	.9762	.9703	.9645	.9587	.9530	.9473	10	19	29	39	49
31	1.9416	.9360	.9304	.9249	.9194	.9139	.9084	.9031	.8977	.8924	9	18	27	36	45
32	1.8871	.8818	.8766	.8714	.8663	.8612	.8561	.8510	.8460	.8410	8	17	25	34	42
33	1.8361	.8312	.8263	.8214	.8166	.8118	.8070	.8023	.7976	.7929	8	16	24	32	40
34	1.7883	.7837	.7791	.7745	.7700	.7655	.7610	.7566	.7522	.7478	7	15	22	30	37
35	1.7434	.7391	.7348	.7305	.7263	.7221	.7179	.7137	.7095	.7054	7	14	21	28	35
36	1.7013	.6972	.6932	.6892	.6852	.6812	.6772	.6733	.6694	.6655	7	13	20	26	33
37	1.6616	.6578	.6540	.6502	.6464	.6427	.6390	.6353	.6316	.6279	6	12	19	25	31
38	1.6243	.6207	.6171	.6135	.6099	.6064	.6029	.5994	.5959	.5925	6	12	18	23	29
39	1.5890	.5856	.5822	.5788	.5755	.5721	.5688	.5655	.5622	.5590	6	11	17	22	28
40	1.5557	.5525	.5493	.5461	.5429	.5398	.5366	.5335	.5304	.5273	5	10	16	21	26
41	1.5243	.5212	.5182	.5151	.5121	.5092	.5062	.5032	.5003	.4974	5	10	15	20	25
42	1.4945	.4916	.4887	.4859	.4830	.4802	.4774	.4746	.4718	.4690	5	9	14	19	23
43	1.4663	.4635	.4608	.4581	.4554	.4527	.4501	.4474	.4448	.4422	4	9	13	18	22
44	1.4396	.4370	.4344	.4318	.4293	.4267	.4242	.4217	.4192	.4167	4	8	13	17	21

Heavy-face type indicates that the integral part is to be decreased by 1.

NATURAL COSECANTS. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	— Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	1.4142	.4118	.4093	.4069	.4044	.4020	.3996	.3972	.3949	.3925	4	8	12	16	20
46	1.3902	.3878	.3855	.3832	.3809	.3786	.3763	.3741	.3718	.3696	4	8	11	15	19
47	1.3673	.3651	.3629	.3607	.3585	.3563	.3542	.3520	.3499	.3478	4	7	11	14	18
48	1.3456	.3435	.3414	.3393	.3373	.3352	.3331	.3311	.3291	.3270	3	7	10	14	17
49	1.3250	.3230	.3210	.3190	.3171	.3151	.3131	.3112	.3093	.3073	3	7	10	13	16
50	1.3054	.3035	.3016	.2997	.2978	.2960	.2941	.2923	.2904	.2886	3	6	9	12	15
51	1.2868	.2849	.2831	.2813	.2796	.2778	.2760	.2742	.2725	.2708	3	6	9	12	15
52	1.2690	.2673	.2656	.2639	.2622	.2605	.2588	.2571	.2554	.2538	3	6	8	11	14
53	1.2521	.2505	.2489	.2472	.2456	.2440	.2424	.2408	.2392	.2376	3	5	8	11	13
54	1.2361	.2345	.2329	.2314	.2299	.2283	.2268	.2253	.2238	.2223	3	5	8	10	13
55	1.2208	.2193	.2178	.2163	.2149	.2134	.2120	.2105	.2091	.2076	2	5	7	10	12
56	1.2062	.2048	.2034	.2020	.2006	.1992	.1978	.1964	.1951	.1937	2	5	7	9	12
57	1.1924	.1910	.1897	.1883	.1870	.1857	.1844	.1831	.1818	.1805	2	4	7	9	11
58	1.1792	.1779	.1766	.1753	.1741	.1728	.1716	.1703	.1691	.1679	2	4	6	8	10
59	1.1666	.1654	.1642	.1630	.1618	.1606	.1594	.1582	.1570	.1559	2	4	6	8	10
60	1.1547	.1535	.1524	.1512	.1501	.1490	.1478	.1467	.1456	.1445	2	4	6	8	9
61	1.1434	.1423	.1412	.1401	.1390	.1379	.1368	.1357	.1347	.1336	2	4	5	7	9
62	1.1326	.1315	.1305	.1294	.1284	.1274	.1264	.1253	.1243	.1233	2	3	5	7	9
63	1.1223	.1213	.1203	.1194	.1184	.1174	.1164	.1155	.1145	.1136	2	3	5	6	8
64	1.1126	.1117	.1107	.1098	.1089	.1079	.1070	.1061	.1052	.1043	2	3	5	6	8
65	1.1034	.1025	.1016	.1007	.0998	.0989	.0981	.0972	.0963	.0955	1	3	4	6	7
66	1.0946	.0938	.0929	.0921	.0913	.0904	.0896	.0888	.0880	.0872	1	3	4	6	7
67	1.0864	.0856	.0848	.0840	.0832	.0824	.0816	.0808	.0801	.0793	1	3	4	5	7
68	1.0785	.0778	.0770	.0763	.0755	.0748	.0740	.0733	.0726	.0719	1	2	4	5	6
69	1.0711	.0704	.0697	.0690	.0683	.0676	.0669	.0662	.0655	.0649	1	2	3	5	6
70	1.0642	.0635	.0628	.0622	.0615	.0608	.0602	.0595	.0589	.0583	1	2	3	4	5
71	1.0576	.0570	.0564	.0557	.0551	.0545	.0539	.0533	.0527	.0521	1	2	3	4	5
72	1.0515	.0509	.0503	.0497	.0491	.0485	.0480	.0474	.0468	.0463	1	2	3	4	5
73	1.0457	.0451	.0446	.0440	.0435	.0429	.0424	.0419	.0413	.0408	1	2	3	4	4
74	1.0403	.0398	.0393	.0388	.0382	.0377	.0372	.0367	.0363	.0358	1	2	2	3	4
75	1.0353	.0348	.0343	.0338	.0334	.0329	.0324	.0320	.0315	.0311	1	2	2	3	4
76	1.0306	.0302	.0297	.0293	.0288	.0284	.0280	.0276	.0271	.0267	1	1	2	3	4
77	1.0263	.0259	.0255	.0251	.0247	.0243	.0239	.0235	.0231	.0227	1	1	2	3	3
78	1.0223	.0220	.0216	.0212	.0209	.0205	.0201	.0198	.0194	.0191	1	1	2	2	3
79	1.0187	.0184	.0180	.0177	.0174	.0170	.0167	.0164	.0161	.0157	1	1	2	2	3
80	1.0154	.0151	.0148	.0145	.0142	.0139	.0136	.0133	.0130	.0127	0	1	1	2	2
81	1.0125	.0122	.0119	.0116	.0114	.0111	.0108	.0106	.0103	.0101	0	1	1	2	2
82	1.0098	.0096	.0093	.0091	.0089	.0086	.0084	.0082	.0079	.0077	0	1	1	2	2
83	1.0075	.0073	.0071	.0069	.0067	.0065	.0063	.0061	.0059	.0057	0	1	1	1	2
84	1.0055	.0053	.0051	.0050	.0048	.0046	.0045	.0043	.0041	.0040	0	1	1	1	1
85	1.0038	.0037	.0035	.0034	.0032	.0031	.0030	.0028	.0027	.0026	0	0	1	1	1
86	1.0024	.0023	.0022	.0021	.0020	.0019	.0018	.0017	.0016	.0015	0	0	1	1	1
87	1.0014	.0013	.0012	.0011	.0010	.0010	.0009	.0008	.0007	.0007	0	0	0	1	1
88	1.0006	.0006	.0005	.0004	.0004	.0003	.0003	.0003	.0002	.0002	0	0	0	0	0
89	1.0002	.0001	.0001	.0001	.0001	.0000	.0000	.0000	.0000	.0000	0	0	0	0	0

The integral part of each of the above cosecants is 1.

LOGARITHMS OF SINES. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	$-\infty$	$\bar{3}$.242	.5429	.7190	.8439	.9408	**.0200**	.0869	.1449	.1961					
1	$\bar{2}$.2419	.2832	.3210	.3558	.3880	.4179	.4459	.4723	.4971	.5206	Use ordinary interpolation.				
2	$\bar{2}$.5428	.5640	.5842	.6035	.6220	.6397	.6567	.6731	.6889	.7041					
3	$\bar{2}$.7188	.7330	.7468	.7602	.7731	.7857	.7979	.8098	.8213	.8326					
4	$\bar{2}$.8436	.8543	.8647	.8749	.8849	.8946	.9042	.9135	.9226	.9315					
5	$\bar{2}$.9403	.9489	.9573	.9655	.9736	.9816	.9894	.9970	**.0046**	.0120	13	26	39	52	65
6	$\bar{1}$.0192	.0264	.0334	.0403	.0472	.0539	.0605	.0670	.0734	.0797	11	22	33	44	55
7	$\bar{1}$.0859	.0920	.0981	.1040	.1099	.1157	.1214	.1271	.1326	.1381	10	19	29	38	48
8	$\bar{1}$.1436	.1489	.1542	.1594	.1646	.1697	.1747	.1797	.1847	.1895	8	17	25	34	42
9	$\bar{1}$.1943	.1991	.2038	.2085	.2131	.2176	.2221	.2266	.2310	.2353	8	15	23	30	38
10	$\bar{1}$.2397	.2439	.2482	.2524	.2565	.2606	.2647	.2687	.2727	.2767	7	14	20	27	34
11	$\bar{1}$.2806	.2845	.2883	.2921	.2959	.2997	.3034	.3070	.3107	.3143	6	12	19	25	31
12	$\bar{1}$.3179	.3214	.3250	.3284	.3319	.3353	.3387	.3421	.3455	.3488	6	11	17	23	28
13	$\bar{1}$.3521	.3554	.3586	.3618	.3650	.3682	.3713	.3745	.3775	.3806	5	11	16	21	26
14	$\bar{1}$.3837	.3867	.3897	.3927	.3957	.3986	.4015	.4044	.4073	.4102	5	10	15	20	24
15	$\bar{1}$.4130	.4158	.4186	.4214	.4242	.4269	.4296	.4323	.4350	.4377	5	9	14	18	23
16	$\bar{1}$.4403	.4430	.4456	.4482	.4508	.4533	.4559	.4584	.4609	.4634	4	9	13	17	21
17	$\bar{1}$.4659	.4684	.4709	.4733	.4757	.4781	.4805	.4829	.4853	.4876	4	8	12	16	20
18	$\bar{1}$.4900	.4923	.4946	.4969	.4992	.5015	.5037	.5060	.5082	.5104	4	8	11	15	19
19	$\bar{1}$.5126	.5148	.5170	.5192	.5213	.5235	.5256	.5278	.5299	.5320	4	7	11	14	18
20	$\bar{1}$.5341	.5361	.5382	.5402	.5423	.5443	.5463	.5484	.5504	.5523	3	7	10	14	17
21	$\bar{1}$.5543	.5563	.5583	.5602	.5621	.5641	.5660	.5679	.5698	.5717	3	6	10	13	16
22	$\bar{1}$.5736	.5754	.5773	.5792	.5810	.5828	.5847	.5865	.5883	.5901	3	6	9	12	15
23	$\bar{1}$.5919	.5937	.5954	.5972	.5990	.6007	.6024	.6042	.6059	.6076	3	6	9	12	15
24	$\bar{1}$.6093	.6110	.6127	.6144	.6161	.6177	.6194	.6210	.6227	.6243	3	6	8	11	14
25	$\bar{1}$.6259	.6276	.6292	.6308	.6324	.6340	.6356	.6371	.6387	.6403	3	5	8	11	13
26	$\bar{1}$.6418	.6434	.6449	.6465	.6480	.6495	.6510	.6526	.6541	.6556	3	5	8	10	13
27	$\bar{1}$.6570	.6585	.6600	.6615	.6629	.6644	.6659	.6673	.6687	.6702	2	5	7	10	12
28	$\bar{1}$.6716	.6730	.6744	.6759	.6773	.6787	.6801	.6814	.6828	.6842	2	5	7	9	12
29	$\bar{1}$.6856	.6869	.6883	.6896	.6910	.6923	.6937	.6950	.6963	.6977	2	4	7	9	11
30	$\bar{1}$.6990	.7003	.7016	.7029	.7042	.7055	.7068	.7080	.7093	.7106	2	4	6	9	11
31	$\bar{1}$.7118	.7131	.7144	.7156	.7168	.7181	.7193	.7205	.7218	.7230	2	4	6	8	10
32	$\bar{1}$.7242	.7254	.7266	.7278	.7290	.7302	.7314	.7326	.7338	.7349	2	4	6	8	10
33	$\bar{1}$.7361	.7373	.7384	.7396	.7407	.7419	.7430	.7442	.7453	.7464	2	4	6	8	10
34	$\bar{1}$.7476	.7487	.7498	.7509	.7520	.7531	.7542	.7553	.7564	.7575	2	4	6	7	9
35	$\bar{1}$.7586	.7597	.7607	.7618	.7629	.7640	.7650	.7661	.7671	.7682	2	4	5	7	9
36	$\bar{1}$.7692	.7703	.7713	.7723	.7734	.7744	.7754	.7764	.7774	.7785	2	3	5	7	9
37	$\bar{1}$.7795	.7805	.7815	.7825	.7835	.7844	.7854	.7864	.7874	.7884	2	3	5	7	8
38	$\bar{1}$.7893	.7903	.7913	.7922	.7932	.7941	.7951	.7960	.7970	.7979	2	3	5	6	8
39	$\bar{1}$.7989	.7998	.8007	.8017	.8026	.8035	.8044	.8053	.8063	.8072	2	3	5	6	8
40	$\bar{1}$.8081	.8090	.8099	.8108	.8117	.8125	.8134	.8143	.8152	.8161	1	3	4	6	7
41	$\bar{1}$.8169	.8178	.8187	.8195	.8204	.8213	.8221	.8230	.8238	.8247	1	3	4	6	7
42	$\bar{1}$.8255	.8264	.8272	.8280	.8289	.8297	.8305	.8313	.8322	.8330	1	3	4	6	7
43	$\bar{1}$.8338	.8346	.8354	.8362	.8370	.8378	.8386	.8394	.8402	.8410	1	3	4	5	7
44	$\bar{1}$.8418	.8426	.8433	.8441	.8449	.8457	.8464	.8472	.8480	.8487	1	3	4	5	6

Heavy-face type indicates that the characteristic is *numerically* 1 less.

LOGARITHMS OF SINES. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	Ī.8495	.8502	.8510	.8517	.8525	.8532	.8540	.8547	.8555	.8562	1	2	4	5	6
46	Ī.8569	.8577	.8584	.8591	.8598	.8606	.8613	.8620	.8627	.8634	1	2	4	5	6
47	Ī.8641	.8648	.8655	.8662	.8669	.8676	.8683	.8690	.8697	.8704	1	2	3	5	6
48	Ī.8711	.8718	.8724	.8731	.8738	.8745	.8751	.8758	.8765	.8771	1	2	3	4	6
49	Ī.8778	.8784	.8791	.8797	.8804	.8810	.8817	.8823	.8830	.8836	1	2	3	4	5
50	Ī.8843	.8849	.8855	.8862	.8868	.8874	.8880	.8887	.8893	.8899	1	2	3	4	5
51	Ī.8905	.8911	.8917	.8923	.8929	.8935	.8941	.8947	.8953	.8959	1	2	3	4	5
52	Ī.8965	.8971	.8977	.8983	.8989	.8995	.9000	.9006	.9012	.9018	1	2	3	4	5
53	Ī.9023	.9029	.9035	.9041	.9046	.9052	.9057	.9063	.9069	.9074	1	2	3	4	5
54	Ī.9080	.9085	.9091	.9096	.9101	.9107	.9112	.9118	.9123	.9128	1	2	3	4	5
55	Ī.9134	.9139	.9144	.9149	.9155	.9160	.9165	.9170	.9175	.9181	1	2	3	3	4
56	Ī.9186	.9191	.9196	.9201	.9206	.9211	.9216	.9221	.9226	.9231	1	2	3	3	4
57	Ī.9236	.9241	.9246	.9251	.9255	.9260	.9265	.9270	.9275	.9279	1	2	2	3	4
58	Ī.9284	.9289	.9294	.9298	.9303	.9308	.9312	.9317	.9322	.9326	1	2	2	3	4
59	Ī.9331	.9335	.9340	.9344	.9349	.9353	.9358	.9362	.9367	.9371	1	1	2	3	4
60	Ī.9375	.9380	.9384	.9388	.9393	.9397	.9401	.9406	.9410	.9414	1	1	2	3	4
61	Ī.9418	.9422	.9427	.9431	.9435	.9439	.9443	.9447	.9451	.9455	1	1	2	3	3
62	Ī.9459	.9463	.9467	.9471	.9475	.9479	.9483	.9487	.9491	.9495	1	1	2	3	3
63	Ī.9499	.9503	.9507	.9510	.9514	.9518	.9522	.9525	.9529	.9533	1	1	2	3	3
64	Ī.9537	.9540	.9544	.9548	.9551	.9555	.9558	.9562	.9566	.9569	1	1	2	2	3
65	Ī.9573	.9576	.9580	.9583	.9587	.9590	.9594	.9597	.9601	.9604	1	1	2	2	3
66	Ī.9607	.9611	.9614	.9617	.9621	.9624	.9627	.9631	.9634	.9637	1	1	2	2	3
67	Ī.9640	.9643	.9647	.9650	.9653	.9656	.9659	.9662	.9666	.9669	1	1	2	2	3
68	Ī.9672	.9675	.9678	.9681	.9684	.9687	.9690	.9693	.9696	.9699	0	1	1	2	2
69	Ī.9702	.9704	.9707	.9710	.9713	.9716	.9719	.9722	.9724	.9727	0	1	1	2	2
70	Ī.9730	.9733	.9735	.9738	.9741	.9743	.9746	.9749	.9751	.9754	0	1	1	2	2
71	Ī.9757	.9759	.9762	.9764	.9767	.9770	.9772	.9775	.9777	.9780	0	1	1	2	2
72	Ī.9782	.9785	.9787	.9789	.9792	.9794	.9797	.9799	.9801	.9804	0	1	1	2	2
73	Ī.9806	.9808	.9811	.9813	.9815	.9817	.9820	.9822	.9824	.9826	0	1	1	2	2
74	Ī.9828	.9831	.9833	.9835	.9837	.9839	.9841	.9843	.9845	.9847	0	1	1	1	2
75	Ī.9849	.9851	.9853	.9855	.9857	.9859	.9861	.9863	.9865	.9867	0	1	1	1	2
76	Ī.9869	.9871	.9873	.9875	.9876	.9878	.9880	.9882	.9884	.9885	0	1	1	1	2
77	Ī.9887	.9889	.9891	.9892	.9894	.9896	.9897	.9899	.9901	.9902	0	1	1	1	1
78	Ī.9904	.9906	.9907	.9909	.9910	.9912	.9913	.9915	.9916	.9918	0	1	1	1	1
79	Ī.9919	.9921	.9922	.9924	.9925	.9927	.9928	.9929	.9931	.9932	0	0	1	1	1
80	Ī.9934	.9935	.9936	.9937	.9939	.9940	.9941	.9943	.9944	.9945	0	0	1	1	1
81	Ī.9946	.9947	.9949	.9950	.9951	.9952	.9953	.9954	.9955	.9956	0	0	1	1	1
82	Ī.9958	.9959	.9960	.9961	.9962	.9963	.9964	.9965	.9966	.9967	0	0	1	1	1
83	Ī.9968	.9968	.9969	.9970	.9971	.9972	.9973	.9974	.9975	.9975	0	0	0	1	1
84	Ī.9976	.9977	.9978	.9978	.9979	.9980	.9981	.9981	.9982	.9983	0	0	0	0	1
85	Ī.9983	.9984	.9985	.9985	.9986	.9987	.9987	.9988	.9988	.9989	0	0	0	0	0
86	Ī.9989	.9990	.9990	.9991	.9991	.9992	.9992	.9993	.9993	.9994	0	0	0	0	0
87	Ī.9994	.9994	.9995	.9995	.9996	.9996	.9996	.9996	.9997	.9997	0	0	0	0	0
88	Ī.9997	.9998	.9998	.9998	.9998	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0
89	Ī.9999	.9999	**.0000**	.0000	.0000	.0000	.0000	.0000	.0000	.0000	0	0	0	0	0

Heavy-face type indicates that the characteristic is *numerically* 1 less.

LOGARITHMS OF COSINES. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	— Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	0.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	**.9999**	0	0	0	0	0
1	Ī.9999	.9999	.9999	.9999	.9999	.9999	.9998	.9998	.9998	.9998	0	0	0	0	0
2	Ī.9997	.9997	.9997	.9997	.9996	.9996	.9996	.9995	.9995	.9994	0	0	0	0	0
3	Ī.9994	.9994	.9993	.9993	.9992	.9992	.9991	.9991	.9990	.9990	0	0	0	0	0
4	Ī.9989	.9989	.9988	.9988	.9987	.9987	.9986	.9985	.9985	.9984	0	0	0	0	0
5	Ī.9983	.9983	.9982	.9981	.9981	.9980	.9979	.9978	.9978	.9977	0	0	0	0	1
6	Ī.9976	.9975	.9975	.9974	.9973	.9972	.9971	.9970	.9969	.9968	0	0	0	1	1
7	Ī.9968	.9967	.9966	.9965	.9964	.9963	.9962	.9961	.9960	.9959	0	0	1	1	1
8	Ī.9958	.9956	.9955	.9954	.9953	.9952	.9951	.9950	.9949	.9947	0	0	1	1	1
9	Ī.9946	.9945	.9944	.9943	.9941	.9940	.9939	.9937	.9936	.9935	0	0	1	1	1
10	Ī.9934	.9932	.9931	.9929	.9928	.9927	.9925	.9924	.9922	.9921	0	0	1	1	1
11	Ī.9919	.9918	.9916	.9915	.9913	.9912	.9910	.9909	.9907	.9906	0	1	1	1	1
12	Ī.9904	.9902	.9901	.9899	.9897	.9896	.9894	.9892	.9891	.9889	0	1	1	1	1
13	Ī.9887	.9885	.9884	.9882	.9880	.9878	.9876	.9875	.9873	.9871	0	1	1	1	2
14	Ī.9869	.9867	.9865	.9863	.9861	.9859	.9857	.9855	.9853	.9851	0	1	1	1	2
15	Ī.9849	.9847	.9845	.9843	.9841	.9839	.9837	.9835	.9833	.9831	0	1	1	1	2
16	Ī.9828	.9826	.9824	.9822	.9820	.9817	.9815	.9813	.9811	.9808	0	1	1	2	2
17	Ī.9806	.9804	.9801	.9799	.9797	.9794	.9792	.9789	.9787	.9785	0	1	1	2	2
18	Ī.9782	.9780	.9777	.9775	.9772	.9770	.9767	.9764	.9762	.9759	0	1	1	2	2
19	Ī.9757	.9754	.9751	.9749	.9746	.9743	.9741	.9738	.9735	.9733	0	1	1	2	2
20	Ī.9730	.9727	.9724	.9722	.9719	.9716	.9713	.9710	.9707	.9704	0	1	1	2	2
21	Ī.9702	.9699	.9696	.9693	.9690	.9687	.9684	.9681	.9678	.9675	0	1	1	2	2
22	Ī.9672	.9669	.9666	.9662	.9659	.9656	.9653	.9650	.9647	.9643	1	1	2	2	3
23	Ī.9640	.9637	.9634	.9631	.9627	.9624	.9621	.9617	.9614	.9611	1	1	2	2	3
24	Ī.9607	.9604	.9601	.9597	.9594	.9590	.9587	.9583	.9580	.9576	1	1	2	2	3
25	Ī.9573	.9569	.9566	.9562	.9558	.9555	.9551	.9548	.9544	.9540	1	1	2	2	3
26	Ī.9537	.9533	.9529	.9525	.9522	.9518	.9514	.9510	.9507	.9503	1	1	2	3	3
27	Ī.9499	.9495	.9491	.9487	.9483	.9479	.9475	.9471	.9467	.9463	1	1	2	3	3
28	Ī.9459	.9455	.9451	.9447	.9443	.9439	.9435	.9431	.9427	.9422	1	1	2	3	3
29	Ī.9418	.9414	.9410	.9406	.9401	.9397	.9393	.9388	.9384	.9380	1	1	2	3	4
30	Ī.9375	.9371	.9367	.9362	.9358	.9353	.9349	.9344	.9340	.9335	1	1	2	3	4
31	Ī.9331	.9326	.9322	.9317	.9312	.9308	.9303	.9298	.9294	.9289	1	2	2	3	4
32	Ī.9284	.9279	.9275	.9270	.9265	.9260	.9255	.9251	.9246	.9241	1	2	2	3	4
33	Ī.9236	.9231	.9226	.9221	.9216	.9211	.9206	.9201	.9196	.9191	1	2	3	3	4
34	Ī.9186	.9181	.9175	.9170	.9165	.9160	.9155	.9149	.9144	.9139	1	2	3	3	4
35	Ī.9134	.9128	.9123	.9118	.9112	.9107	.9101	.9096	.9091	.9085	1	2	3	4	5
36	Ī.9080	.9074	.9069	.9063	.9057	.9052	.9046	.9041	.9035	.9029	1	2	3	4	5
37	Ī.9023	.9018	.9012	.9006	.9000	.8995	.8989	.8983	.8977	.8971	1	2	3	4	5
38	Ī.8965	.8959	.8953	.8947	.8941	.8935	.8929	.8923	.8917	.8911	1	2	3	4	5
39	Ī.8905	.8899	.8893	.8887	.8880	.8874	.8868	.8862	.8855	.8849	1	2	3	4	5
40	Ī.8843	.8836	.8830	.8823	.8817	.8810	.8804	.8797	.8791	.8784	1	2	3	4	5
41	Ī.8778	.8771	.8765	.8758	.8751	.8745	.8738	.8731	.8724	.8718	1	2	3	5	6
42	Ī.8711	.8704	.8697	.8690	.8683	.8676	.8669	.8662	.8655	.8648	1	2	3	5	6
43	Ī.8641	.8634	.8627	.8620	.8613	.8606	.8598	.8591	.8584	.8577	1	2	4	5	6
44	Ī.8569	.8562	.8555	.8547	.8540	.8532	.8525	.8517	.8510	.8502	1	2	4	5	6

Heavy-face type indicates that the characteristic is *numerically* 1 greater.

LOGARITHMS OF COSINES. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	– Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	$\bar{1}$.8495	.8487	.8480	.8472	.8464	.8457	.8449	.8441	.8433	.8426	1	3	4	5	6
46	$\bar{1}$.8418	.8410	.8402	.8394	.8386	.8378	.8370	.8362	.8354	.8346	1	3	4	5	7
47	$\bar{1}$.8338	.8330	.8322	.8313	.8305	.8297	.8289	.8280	.8272	.8264	1	3	4	6	7
48	$\bar{1}$.8255	.8247	.8238	.8230	.8221	.8213	.8204	.8195	.8187	.8178	1	3	4	6	7
49	$\bar{1}$.8169	.8161	.8152	.8143	.8134	.8125	.8117	.8108	.8099	.8090	1	3	4	6	7
50	$\bar{1}$.8081	.8072	.8063	.8053	.8044	.8035	.8026	.8017	.8007	.7998	2	3	5	6	8
51	$\bar{1}$.7989	.7979	.7970	.7960	.7951	.7941	.7932	.7922	.7913	.7903	2	3	5	6	8
52	$\bar{1}$.7893	.7884	.7874	.7864	.7854	.7844	.7835	.7825	.7815	.7805	2	3	5	7	8
53	$\bar{1}$.7795	.7785	.7774	.7764	.7754	.7744	.7734	.7723	.7713	.7703	2	3	5	7	9
54	$\bar{1}$.7692	.7682	.7671	.7661	.7650	.7640	.7629	.7618	.7607	.7597	2	4	5	7	9
55	$\bar{1}$.7586	.7575	.7564	.7553	.7542	.7531	.7520	.7509	.7498	.7487	2	4	6	7	9
56	$\bar{1}$.7476	.7464	.7453	.7442	.7430	.7419	.7407	.7396	.7384	.7373	2	4	6	8	10
57	$\bar{1}$.7361	.7349	.7338	.7326	.7314	.7302	.7290	.7278	.7266	.7254	2	4	6	8	10
58	$\bar{1}$.7242	.7230	.7218	.7205	.7193	.7181	.7168	.7156	.7144	.7131	2	4	6	8	10
59	$\bar{1}$.7118	.7106	.7093	.7080	.7068	.7055	.7042	.7029	.7016	.7003	2	4	6	9	11
60	$\bar{1}$.6990	.6977	.6963	.6950	.6937	.6923	.6910	.6896	.6883	.6869	2	4	7	9	11
61	$\bar{1}$.6856	.6842	.6828	.6814	.6801	.6787	.6773	.6759	.6744	.6730	2	5	7	9	12
62	$\bar{1}$.6716	.6702	.6687	.6673	.6659	.6644	.6629	.6615	.6600	.6585	2	5	7	10	12
63	$\bar{1}$.6570	.6556	.6541	.6526	.6510	.6495	.6480	.6465	.6449	.6434	3	5	8	10	13
64	$\bar{1}$.6418	.6403	.6387	.6371	.6356	.6340	.6324	.6308	.6292	.6276	3	5	8	11	13
65	$\bar{1}$.6259	.6243	.6227	.6210	.6194	.6177	.6161	.6144	.6127	.6110	3	6	8	11	14
66	$\bar{1}$.6093	.6076	.6059	.6042	.6024	.6007	.5990	.5972	.5954	.5937	3	6	9	12	15
67	$\bar{1}$.5919	.5901	.5883	.5865	.5847	.5828	.5810	.5792	.5773	.5754	3	6	9	12	15
68	$\bar{1}$.5736	.5717	.5698	.5679	.5660	.5641	.5621	.5602	.5583	.5563	3	6	10	13	16
69	$\bar{1}$.5543	.5523	.5504	.5484	.5463	.5443	.5423	.5402	.5382	.5361	3	7	10	14	17
70	$\bar{1}$.5341	.5320	.5299	.5278	.5256	.5235	.5213	.5192	.5170	.5148	4	7	11	14	18
71	$\bar{1}$.5126	.5104	.5082	.5060	.5037	.5015	.4992	.4969	.4946	.4923	4	8	11	15	19
72	$\bar{1}$.4900	.4876	.4853	.4829	.4805	.4781	.4757	.4733	.4709	.4684	4	8	12	16	20
73	$\bar{1}$.4659	.4634	.4609	.4584	.4559	.4533	.4508	.4482	.4456	.4430	4	9	13	17	21
74	$\bar{1}$.4403	.4377	.4350	.4323	.4296	.4269	.4242	.4214	.4186	.4158	5	9	14	18	23
75	$\bar{1}$.4130	.4102	.4073	.4044	.4015	.3986	.3957	.3927	.3897	.3867	5	10	15	20	24
76	$\bar{1}$.3837	.3806	.3775	.3745	.3713	.3682	.3650	.3618	.3586	.3554	5	11	16	21	26
77	$\bar{1}$.3521	.3488	.3455	.3421	.3387	.3353	.3319	.3284	.3250	.3214	6	11	17	23	28
78	$\bar{1}$.3179	.3143	.3107	.3070	.3034	.2997	.2959	.2921	.2883	.2845	6	12	19	25	31
79	$\bar{1}$.2806	.2767	.2727	.2687	.2647	.2606	.2565	.2524	.2482	.2439	7	14	20	27	34
80	$\bar{1}$.2397	.2353	.2310	.2266	.2221	.2176	.2131	.2085	.2038	.1991	8	15	23	30	38
81	$\bar{1}$.1943	.1895	.1847	.1797	.1747	.1697	.1646	.1594	.1542	.1489	8	17	25	34	42
82	$\bar{1}$.1436	.1381	.1326	.1271	.1214	.1157	.1099	.1040	.0981	.0920	10	19	29	38	48
83	$\bar{1}$.0859	.0797	.0734	.0670	.0605	.0539	.0472	.0403	.0334	.0264	11	22	33	44	55
84	$\bar{1}$.0192	.0120	.0046	**.9970**	.9894	.9816	.9736	.9655	.9573	.9489	13	26	39	52	66
85	$\bar{2}$.9403	.9315	.9226	.9135	.9042	.8946	.8849	.8749	.8647	.8543	16	32	48	64	80
86	$\bar{2}$.8436	.8326	.8213	.8098	.7979	.7857	.7731	.7602	.7468	.7330	Use ordinary interpolation.				
87	$\bar{2}$.7188	.7041	.6889	.6731	.6567	.6397	.6220	.6035	.5842	.5640					
88	$\bar{2}$.5428	.5206	.4971	.4723	.4459	.4179	.3880	.3558	.3210	.2832					
89	$\bar{2}$.2419	.1961	.1449	.0869	.0200	**.9408**	.8439	.7190	.5429	.2419					

Heavy-face type indicates that the characteristic is *numerically* 1 greater.

LOGARITHMS OF TANGENTS. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	$-\infty$	$\bar{3}$.242	.5429	.7190	.8439	.9408	**.0200**	.0870	.1450	.1962					
1	$\bar{2}$.2419	.2833	.3211	.3559	.3881	.4181	.4461	.4725	.4973	.5208	Use ordinary interpolation.				
2	$\bar{2}$.5431	.5643	.5845	.6038	.6223	.6401	.6571	.6736	.6894	.7046					
3	$\bar{2}$.7194	.7337	.7475	.7609	.7739	.7865	.7988	.8107	.8223	.8336					
4	$\bar{2}$.8446	.8554	.8659	.8762	.8862	.8960	.9056	.9150	.9241	.9331	16	32	48	64	81
5	$\bar{2}$.9420	.9506	.9591	.9674	.9756	.9836	.9915	.9992	**.0068**	.0143	13	26	40	53	66
6	$\bar{1}$.0216	.0289	.0360	.0430	.0499	.0567	.0633	.0699	.0764	.0828	11	22	34	45	56
7	$\bar{1}$.0891	.0954	.1015	.1076	.1135	.1194	.1252	.1310	.1367	.1423	10	20	29	39	49
8	$\bar{1}$.1478	.1533	.1587	.1640	.1693	.1745	.1797	.1848	.1898	.1948	9	17	26	35	43
9	$\bar{1}$.1997	.2046	.2094	.2142	.2189	.2236	.2282	.2328	.2374	.2419	8	16	23	31	39
10	$\bar{1}$.2463	.2507	.2551	.2594	.2637	.2680	.2722	.2764	.2805	.2846	7	14	21	28	35
11	$\bar{1}$.2887	.2927	.2967	.3006	.3046	.3085	.3123	.3162	.3200	.3237	6	13	19	26	32
12	$\bar{1}$.3275	.3312	.3349	.3385	.3422	.3458	.3493	.3529	.3564	.3599	6	12	18	24	30
13	$\bar{1}$.3634	.3668	.3702	.3736	.3770	.3804	.3837	.3870	.3903	.3935	6	11	17	22	28
14	$\bar{1}$.3968	.4000	.4032	.4064	.4095	.4127	.4158	.4189	.4220	.4250	5	10	16	21	26
15	$\bar{1}$.4281	.4311	.4341	.4371	.4400	.4430	.4459	.4488	.4517	.4546	5	10	15	20	25
16	$\bar{1}$.4575	.4603	.4632	.4660	.4688	.4716	.4744	.4771	.4799	.4826	5	9	14	19	23
17	$\bar{1}$.4853	.4880	.4907	.4934	.4961	.4987	.5014	.5040	.5066	.5092	4	9	13	18	22
18	$\bar{1}$.5118	.5143	.5169	.5195	.5220	.5245	.5270	.5295	.5320	.5345	4	8	13	17	21
19	$\bar{1}$.5370	.5394	.5419	.5443	.5467	.5491	.5516	.5539	.5563	.5587	4	8	12	16	20
20	$\bar{1}$.5611	.5634	.5658	.5681	.5704	.5727	.5750	.5773	.5796	.5819	4	8	12	15	19
21	$\bar{1}$.5842	.5864	.5887	.5909	.5932	.5954	.5976	.5998	.6020	.6042	4	7	11	15	19
22	$\bar{1}$.6064	.6086	.6108	.6129	.6151	.6172	.6194	.6215	.6236	.6257	4	7	11	14	18
23	$\bar{1}$.6279	.6300	.6321	.6341	.6362	.6383	.6404	.6424	.6445	.6465	3	7	10	14	17
24	$\bar{1}$.6486	.6506	.6527	.6547	.6567	.6587	.6607	.6627	.6647	.6667	3	7	10	13	17
25	$\bar{1}$.6687	.6706	.6726	.6746	.6765	.6785	.6804	.6824	.6843	.6863	3	7	10	13	16
26	$\bar{1}$.6882	.6901	.6920	.6939	.6958	.6977	.6996	.7015	.7034	.7053	3	6	9	13	16
27	$\bar{1}$.7072	.7090	.7109	.7128	.7146	.7165	.7183	.7202	.7220	.7238	3	6	9	12	15
28	$\bar{1}$.7257	.7275	.7293	.7311	.7330	.7348	.7366	.7384	.7402	.7420	3	6	9	12	15
29	$\bar{1}$.7438	.7455	.7473	.7491	.7509	.7526	.7544	.7562	.7579	.7597	3	6	9	12	15
30	$\bar{1}$.7614	.7632	.7649	.7667	.7684	.7701	.7719	.7736	.7753	.7771	3	6	9	12	14
31	$\bar{1}$.7788	.7805	.7822	.7839	.7856	.7873	.7890	.7907	.7924	.7941	3	6	9	11	14
32	$\bar{1}$.7958	.7975	.7992	.8008	.8025	.8042	.8059	.8075	.8092	.8109	3	6	8	11	14
33	$\bar{1}$.8125	.8142	.8158	.8175	.8191	.8208	.8224	.8241	.8257	.8274	3	5	8	11	14
34	$\bar{1}$.8290	.8306	.8323	.8339	.8355	.8371	.8388	.8404	.8420	.8436	3	5	8	11	14
35	$\bar{1}$.8452	.8468	.8484	.8501	.8517	.8533	.8549	.8565	.8581	.8597	3	5	8	11	13
36	$\bar{1}$.8613	.8629	.8644	.8660	.8676	.8692	.8708	.8724	.8740	.8755	3	5	8	11	13
37	$\bar{1}$.8771	.8787	.8803	.8818	.8834	.8850	.8865	.8881	.8897	.8912	3	5	8	10	13
38	$\bar{1}$.8928	.8944	.8959	.8975	.8990	.9006	.9022	.9037	.9053	.9068	3	5	8	10	13
39	$\bar{1}$.9084	.9099	.9115	.9130	.9146	.9161	.9176	.9192	.9207	.9223	3	5	8	10	13
40	$\bar{1}$.9238	.9254	.9269	.9284	.9300	.9315	.9330	.9346	.9361	.9376	3	5	8	10	13
41	$\bar{1}$.9392	.9407	.9422	.9438	.9453	.9468	.9483	.9499	.9514	.9529	3	5	8	10	13
42	$\bar{1}$.9544	.9560	.9575	.9590	.9605	.9621	.9636	.9651	.9666	.9681	3	5	8	10	13
43	$\bar{1}$.9697	.9712	.9727	.9742	.9757	.9773	.9788	.9803	.9818	.9833	3	5	8	10	13
44	$\bar{1}$.9848	.9864	.9879	.9894	.9909	.9924	.9939	.9955	.9970	.9985	3	5	8	10	13

Heavy-face type indicates that the characteristic is *numerically* 1 less.

LOGARITHMS OF TANGENTS. 45°–90

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	0.0000	.0015	.0030	.0045	.0061	.0076	.0091	.0106	.0121	.0136	3	5	8	10	13
46	0.0152	.0167	.0182	.0197	.0212	.0228	.0243	.0258	.0273	.0288	3	5	8	10	13
47	0.0303	.0319	.0334	.0349	.0364	.0379	.0395	.0410	.0425	.0440	3	5	8	10	13
48	0.0456	.0471	.0486	.0501	.0517	.0532	.0547	.0562	.0578	.0593	3	5	8	10	13
49	0.0608	.0624	.0639	.0654	.0670	.0685	.0700	.0716	.0731	.0746	3	5	8	10	13
50	0.0762	.0777	.0793	.0808	.0824	.0839	.0854	.0870	.0885	.0901	3	5	8	10	13
51	0.0916	.0932	.0947	.0963	.0978	.0994	.1010	.1025	.1041	.1056	3	5	8	10	13
52	0.1072	.1088	.1103	.1119	.1135	.1150	.1166	.1182	.1197	.1213	3	5	8	10	13
53	0.1229	.1245	.1260	.1276	.1292	.1308	.1324	.1340	.1356	.1371	3	5	8	11	13
54	0.1387	.1403	.1419	.1435	.1451	.1467	.1483	.1499	.1516	.1532	3	5	8	11	13
55	0.1548	.1564	.1580	.1596	.1612	.1629	.1645	.1661	.1677	.1694	3	5	8	11	14
56	0.1710	.1726	.1743	.1759	.1776	.1792	.1809	.1825	.1842	.1858	3	5	8	11	14
57	0.1875	.1891	.1908	.1925	.1941	.1958	.1975	.1992	.2008	.2025	3	6	8	11	14
58	0.2042	.2059	.2076	.2093	.2110	.2127	.2144	.2161	.2178	.2195	3	6	9	11	14
59	0.2212	.2229	.2247	.2264	.2281	.2299	.2316	.2333	.2351	.2368	3	6	9	12	14
60	0.2386	.2403	.2421	.2438	.2456	.2474	.2491	.2509	.2527	.2545	3	6	9	12	15
61	0.2562	.2580	.2598	.2616	.2634	.2652	.2670	.2689	.2707	.2725	3	6	9	12	15
62	0.2743	.2762	.2780	.2798	.2817	.2835	.2854	.2872	.2891	.2910	3	6	9	12	15
63	0.2928	.2947	.2966	.2985	.3004	.3023	.3042	.3061	.3080	.3099	3	6	9	13	16
64	0.3118	.3137	.3157	.3176	.3196	.3215	.3235	.3254	.3274	.3294	3	7	10	13	16
65	0.3313	.3333	.3353	.3373	.3393	.3413	.3433	.3453	.3473	.3494	3	7	10	13	17
66	0.3514	.3535	.3555	.3576	.3596	.3617	.3638	.3659	.3679	.3700	3	7	10	14	17
67	0.3721	.3743	.3764	.3785	.3806	.3828	.3849	.3871	.3892	.3914	4	7	11	14	18
68	0.3936	.3958	.3980	.4002	.4024	.4046	.4068	.4091	.4113	.4136	4	7	11	15	19
69	0.4158	.4181	.4204	.4227	.4250	.4273	.4296	.4319	.4342	.4366	4	8	12	15	19
70	0.4389	.4413	.4437	.4461	.4484	.4509	.4533	.4557	.4581	.4606	4	8	12	16	20
71	0.4630	.4655	.4680	.4705	.4730	.4755	.4780	.4805	.4831	.4857	4	8	13	17	21
72	0.4882	.4908	.4934	.4960	.4986	.5013	.5039	.5066	.5093	.5120	4	9	13	18	22
73	0.5147	.5174	.5201	.5229	.5256	.5284	.5312	.5340	.5368	.5397	5	9	14	19	23
74	0.5425	.5454	.5483	.5512	.5541	.5570	.5600	.5629	.5659	.5689	5	10	15	20	25
75	0.5719	.5750	.5780	.5811	.5842	.5873	.5905	.5936	.5968	.6000	5	10	16	21	26
76	0.6032	.6065	.6097	.6130	.6163	.6196	.6230	.6264	.6298	.6332	6	11	17	22	28
77	0.6366	.6401	.6436	.6471	.6507	.6542	.6578	.6615	.6651	.6688	6	12	18	24	30
78	0.6725	.6763	.6800	.6838	.6877	.6915	.6954	.6994	.7033	.7073	6	13	19	26	32
79	0.7113	.7154	.7195	.7236	.7278	.7320	.7363	.7406	.7449	.7493	7	14	21	28	35
80	0.7537	.7581	.7626	.7672	.7718	.7764	.7811	.7858	.7906	.7954	8	16	21	31	39
81	0.8003	.8052	.8102	.8152	.8203	.8255	.8307	.8360	.8413	.8467	9	17	26	35	43
82	0.8522	.8577	.8633	.8690	.8748	.8806	.8865	.8924	.8985	.9046	10	20	29	39	49
83	0.9109	.9172	.9236	.9301	.9367	.9433	.9501	.9570	.9640	.9711	11	22	34	45	56
84	0.9784	.9857	.9932	**.0008**	.0085	.0164	.0244	.0326	.0409	.0494	13	26	40	53	66
85	1.0580	.0669	.0759	.0850	.0944	.1040	.1138	.1238	.1341	.1446	16	32	48	65	81
86	1.1554	.1664	.1777	.1893	.2012	.2135	.2261	.2391	.2525	.2663	Use ordinary interpolation.				
87	1.2806	.2954	.3106	.3264	.3429	.3599	.3777	.3962	.4155	.4357					
88	1.4569	.4792	.5027	.5275	.5539	.5819	.6119	.6441	.6789	.7167					
89	1.7581	.8038	.8550	.9130	.9800	**.0592**	.1561	.2810	.4571	.7581					

Heavy-face type indicates that the characteristic is 1 greater than before.

LOGARITHMS OF COTANGENTS. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	— Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	∞	2.758	.4571	.2810	.1561	.0592	**.9800**	.9130	.8550	.8038					
1	1.7581	.7167	.6789	.6441	.6119	.5819	.5539	.5275	.5027	.4792	Use ordinary interpolation.				
2	1.4569	.4357	.4155	.3962	.3777	.3599	.3429	.3264	.3106	.2954					
3	1.2806	.2663	.2525	.2391	.2261	.2135	.2012	.1893	.1777	.1664					
4	1.1554	.1446	.1341	.1238	.1138	.1040	.0944	.0850	.0759	.0669	16	32	48	65	81
5	1.0580	.0494	.0409	.0326	.0244	.0164	.0085	.0008	**.9932**	.9857	13	26	40	53	66
6	0.9784	.9711	.9640	.9570	.9501	.9433	.9367	.9301	.9236	.9172	11	22	34	45	56
7	0.9109	.9046	.8985	.8924	.8865	.8806	.8748	.8690	.8633	.8577	10	20	29	39	49
8	0.8522	.8467	.8413	.8360	.8307	.8255	.8203	.8152	.8102	.8052	9	17	26	35	43
9	0.8003	.7954	.7906	.7858	.7811	.7764	.7718	.7672	.7626	.7581	8	16	21	31	39
10	0.7537	.7493	.7449	.7406	.7363	.7320	.7278	.7236	.7195	.7154	7	14	21	28	35
11	0.7113	.7073	.7033	.6994	.6954	.6915	.6877	.6838	.6800	.6763	6	13	19	26	32
12	0.6725	.6688	.6651	.6615	.6578	.6542	.6507	.6471	.6436	.6401	6	12	18	24	30
13	0.6366	.6332	.6298	.6264	.6230	.6196	.6163	.6130	.6097	.6065	6	11	17	22	28
14	0.6032	.6000	.5968	.5936	.5905	.5873	.5842	.5811	.5780	.5750	5	10	16	21	26
15	0.5719	.5689	.5659	.5629	.5600	.5570	.5541	.5512	.5483	.5454	5	10	15	20	25
16	0.5425	.5397	.5368	.5340	.5312	.5284	.5256	.5229	.5201	.5174	5	9	14	19	23
17	0.5147	.5120	.5093	.5066	.5039	.5013	.4986	.4960	.4934	.4908	4	9	13	18	22
18	0.4882	.4857	.4831	.4805	.4780	.4755	.4730	.4705	.4680	.4655	4	8	13	17	21
19	0.4630	.4606	.4581	.4557	.4533	.4509	.4484	.4461	.4437	.4413	4	8	12	16	20
20	0.4389	.4366	.4342	.4319	.4296	.4273	.4250	.4227	.4204	.4181	4	8	12	15	19
21	0.4158	.4136	.4113	.4091	.4068	.4046	.4024	.4002	.3980	.3958	4	7	11	15	19
22	0.3936	.3914	.3892	.3871	.3849	.3828	.3806	.3785	.3764	.3743	4	7	11	14	18
23	0.3721	.3700	.3679	.3659	.3638	.3617	.3596	.3576	.3555	.3535	3	7	10	14	17
24	0.3514	.3494	.3473	.3453	.3433	.3413	.3393	.3373	.3353	.3333	3	7	10	13	17
25	0.3313	.3294	.3274	.3254	.3235	.3215	.3196	.3176	.3157	.3137	3	7	10	13	16
26	0.3118	.3099	.3080	.3061	.3042	.3023	.3004	.2985	.2966	.2947	3	6	9	13	16
27	0.2928	.2910	.2891	.2872	.2854	.2835	.2817	.2798	.2780	.2762	3	6	9	12	15
28	0.2743	.2725	.2707	.2689	.2670	.2652	.2634	.2616	.2598	.2580	3	6	9	12	15
29	0.2562	.2545	.2527	.2509	.2491	.2474	.2456	.2438	.2421	.2403	3	6	9	12	15
30	0.2386	.2368	.2351	.2333	.2316	.2299	.2281	.2264	.2247	.2229	3	6	9	12	14
31	0.2212	.2195	.2178	.2161	.2144	.2127	.2110	.2093	.2076	.2059	3	6	9	11	14
32	0.2042	.2025	.2008	.1992	.1975	.1958	.1941	.1925	.1908	.1891	3	6	8	11	14
33	0.1875	.1858	.1842	.1825	.1809	.1792	.1776	.1759	.1743	.1726	3	5	8	11	14
34	0.1710	.1694	.1677	.1661	.1645	.1629	.1612	.1596	.1580	.1564	3	5	8	11	14
35	0.1548	.1532	.1516	.1499	.1483	.1467	.1451	.1435	.1419	.1403	3	5	8	11	13
36	0.1387	.1371	.1356	.1340	.1324	.1308	.1292	.1276	.1260	.1245	3	5	8	11	13
37	0.1229	.1213	.1197	.1182	.1166	.1150	.1135	.1119	.1103	.1088	3	5	8	10	13
38	0.1072	.1056	.1041	.1025	.1010	.0994	.0978	.0963	.0947	.0932	3	5	8	10	13
39	0.0916	.0901	.0885	.0870	.0854	.0839	.0824	.0808	.0793	.0777	3	5	8	10	13
40	0.0762	.0746	.0731	.0716	.0700	.0685	.0670	.0654	.0639	.0624	3	5	8	10	13
41	0.0608	.0593	.0578	.0562	.0547	.0532	.0517	.0501	.0486	.0471	3	5	8	10	13
42	0.0456	.0440	.0425	.0410	.0395	.0379	.0364	.0349	.0334	.0319	3	5	8	10	13
43	0.0303	.0288	.0273	.0258	.0243	.0228	.0212	.0197	.0182	.0167	3	5	8	10	13
44	0.0152	.0136	.0121	.0106	.0091	.0076	.0061	.0045	.0030	.0015	3	5	8	10	13

Heavy-face type indicates that the characteristic is *numerically* 1 less.

LOGARITHMS OF COTANGENTS. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	– Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	0.0000	**.9985**	.9970	.9955	.9939	.9924	.9909	.9894	.9879	.9864	3	5	8	10	13
46	$\bar{1}$.9848	.9833	.9818	.9803	.9788	.9773	.9757	.9742	.9727	.9712	3	5	8	10	13
47	$\bar{1}$.9697	.9681	.9666	.9651	.9636	.9621	.9605	.9590	.9575	.9560	3	5	8	10	13
48	$\bar{1}$.9544	.9529	.9514	.9499	.9483	.9468	.9453	.9438	.9422	.9407	3	5	8	10	13
49	$\bar{1}$.9392	.9376	.9361	.9346	.9330	.9315	.9300	.9284	.9269	.9254	3	5	8	10	13
50	$\bar{1}$.9238	.9223	.9207	.9192	.9176	.9161	.9146	.9130	.9115	.9099	3	5	8	10	13
51	$\bar{1}$.9084	.9068	.9053	.9037	.9022	.9006	.8990	.8975	.8959	.8944	3	5	8	10	13
52	$\bar{1}$.8928	.8912	.8897	.8881	.8865	.8850	.8834	.8818	.8803	.8787	3	5	8	10	13
53	$\bar{1}$.8771	.8755	.8740	.8724	.8708	.8692	.8676	.8660	.8644	.8629	3	5	8	11	13
54	$\bar{1}$.8613	.8597	.8581	.8565	.8549	.8533	.8517	.8501	.8484	.8468	3	5	8	11	13
55	$\bar{1}$.8452	.8436	.8420	.8404	.8388	.8371	.8355	.8339	.8323	.8306	3	5	8	11	14
56	$\bar{1}$.8290	.8274	.8257	.8241	.8224	.8208	.8191	.8175	.8158	.8142	3	5	8	11	14
57	$\bar{1}$.8125	.8109	.8092	.8075	.8059	.8042	.8025	.8008	.7992	.7975	3	6	8	11	14
58	$\bar{1}$.7958	.7941	.7924	.7907	.7890	.7873	.7856	.7839	.7822	.7805	3	6	9	11	14
59	$\bar{1}$.7788	.7771	.7753	.7736	.7719	.7701	.7684	.7667	.7649	.7632	3	6	9	12	14
60	$\bar{1}$.7614	.7597	.7579	.7562	.7544	.7526	.7509	.7491	.7473	.7455	3	6	9	12	15
61	$\bar{1}$.7438	.7420	.7402	.7384	.7366	.7348	.7330	.7311	.7293	.7275	3	6	9	12	15
62	$\bar{1}$.7257	.7238	.7220	.7202	.7183	.7165	.7146	.7128	.7109	.7090	3	6	9	12	15
63	$\bar{1}$.7072	.7053	.7034	.7015	.6996	.6977	.6958	.6939	.6920	.6901	3	6	9	13	16
64	$\bar{1}$.6882	.6863	.6843	.6824	.6804	.6785	.6765	.6746	.6726	.6706	3	7	10	13	16
65	$\bar{1}$.6687	.6667	.6647	.6627	.6607	.6587	.6567	.6547	.6527	.6506	3	7	10	13	17
66	$\bar{1}$.6486	.6465	.6445	.6424	.6404	.6383	.6362	.6341	.6321	.6300	3	7	10	14	17
67	$\bar{1}$.6279	.6257	.6236	.6215	.6194	.6172	.6151	.6129	.6108	.6086	4	7	11	14	18
68	$\bar{1}$.6064	.6042	.6020	.5998	.5976	.5954	.5932	.5909	.5887	.5864	4	7	11	15	19
69	$\bar{1}$.5842	.5819	.5796	.5773	.5750	.5727	.5704	.5681	.5658	.5634	4	8	12	15	19
70	$\bar{1}$.5611	.5587	.5563	.5539	.5516	.5491	.5467	.5443	.5419	.5394	4	8	12	16	20
71	$\bar{1}$.5370	.5345	.5320	.5295	.5270	.5245	.5220	.5195	.5169	.5143	4	8	13	17	21
72	$\bar{1}$.5118	.5092	.5066	.5040	.5014	.4987	.4961	.4934	.4907	.4880	4	9	13	18	22
73	$\bar{1}$.4853	.4826	.4799	.4771	.4744	.4716	.4688	.4660	.4632	.4603	5	9	14	19	23
74	$\bar{1}$.4575	.4546	.4517	.4488	.4459	.4430	.4400	.4371	.4341	.4311	5	10	15	20	25
75	$\bar{1}$.4281	.4250	.4220	.4189	.4158	.4127	.4095	.4064	.4032	.4000	5	10	16	21	26
76	$\bar{1}$.3968	.3935	.3903	.3870	.3837	.3804	.3770	.3736	.3702	.3668	6	11	17	22	28
77	$\bar{1}$.3634	.3599	.3564	.3529	.3493	.3458	.3422	.3385	.3349	.3312	6	12	18	24	30
78	$\bar{1}$.3275	.3237	.3200	.3162	.3123	.3085	.3046	.3006	.2967	.2927	6	13	19	26	32
79	$\bar{1}$.2887	.2846	.2805	.2764	.2722	.2680	.2637	.2594	.2551	.2507	7	14	21	28	35
80	$\bar{1}$.2463	.2419	.2374	.2328	.2282	.2236	.2189	.2142	.2094	.2046	8	16	23	31	39
81	$\bar{1}$.1997	.1948	.1898	.1848	.1797	.1745	.1693	.1640	.1587	.1533	9	17	26	35	43
82	$\bar{1}$.1478	.1423	.1367	.1310	.1252	.1194	.1135	.1076	.1015	.0954	10	20	29	39	49
83	$\bar{1}$.0891	.0828	.0764	.0699	.0633	.0567	.0499	.0430	.0360	.0289	11	22	34	45	56
84	$\bar{1}$.0216	.0143	.0068	**.9992**	.9915	.9836	.9756	.9674	.9591	.9506	13	26	40	53	66
85	$\bar{2}$.9420	.9331	.9241	.9150	.9056	.8960	.8862	.8762	.8659	.8554	16	32	48	64	81
86	$\bar{2}$.8446	.8336	.8223	.8107	.7988	.7865	.7739	.7609	.7475	.7337					
87	$\bar{2}$.7194	.7046	.6894	.6736	.6571	.6401	.6223	.6038	.5845	.5643	Use ordinary interpolation.				
88	$\bar{2}$.5431	.5208	.4973	.4725	.4461	.4181	.3881	.3559	.3211	.2833					
89	$\bar{2}$.2419	.1962	.1450	.0870	.0200	**.9408**	.8439	.7190	.5429	.2419					

Heavy-face type indicates that the characteristic is *numerically* 1 greater.

4. Common Measures

Length

12 inches (in.) = 1 foot (ft.)
3 feet = 1 yard (yd.)
$5\frac{1}{2}$ yards, or $16\frac{1}{2}$ feet = 1 rod (rd.)
320 rods, or 5280 feet = 1 mile (mi.)

Square Measure

144 square inches (sq. in.) = 1 square foot (sq. ft.)
9 square feet = 1 square yard (sq. yd.)
$30\frac{1}{4}$ square yards = 1 square rod (sq. rd.)
160 square rods = 1 acre (A.)
640 acres = 1 square mile (sq. mi.)

Cubic Measure

1728 cubic inches (cu. in.) = 1 cubic foot (cu. ft.)
27 cubic feet = 1 cubic yard (cu. yd.)

Metric Length

1 millimeter (mm.) = 0.001 meter (m.)
1 centimeter (cm.) = 0.01 meter
1 kilometer (km.) = 1000 meters

For ordinary purposes 1 cm. may be taken as equivalent to 0.4 in.; 1 m. to 39.37 in., or 1.1 yd.; and 1 km. to 0.62 mi.

Metric Weight

1000 grams (g.) = 1 kilogram (kg.)
1000 kilograms = 1 metric ton (t.)

A gram is approximately equivalent to 0.035 oz. in our common system, and 1 kg. is approximately equivalent to 2.2 lb.

Metric Capacity

100 liters (l.) = 1 hektoliter (hl.)

The liter is the volume of a cube that is 0.1 m. on an edge; that is, 0.001 cu. m. contains 1 l. The liter is practically equivalent to 1 qt. in our common system.

INDEX

PAGE

Abscissa 37, 85, 163
Acceleration 248
Accuracy, of logarithms . . . 71
in measuring angles 90
of results 16, 21, 91
Algebra 1
Angles, functions of 86
generation of 83
magnetic 139
measurement of 90
positive and negative 83
trisection of 228
Antilogarithm 75
Area, between two curves . . . 311
under a curve 306
Heron's formula for 152
rate of change of 307
rectangle method of 301
Simpson's rule for 338
trapezoidal method of . . . 302
of a triangle 129, 151
Asymptote 220, 223, 224
Axes 36, 163
Axis, conjugate 219
of an ellipse 213
of a function 38
of a hyperbola 219
of a parabola 205
of reference 36
transverse 219

Base of a logarithm 65
Binomial Theorem 53
Boscovich 215, 221

Calculus 239
Cartesian coordinates 166
Center, of an ellipse 213
of a hyperbola 219

PAGE

Change, in area 307
in trigonometric functions . . 106
in variables 244, 289
Characteristic 68
Check for roots 23
Chord, focal 205
Circle, equation of the . . 178, 196
as a locus 178, 195
unit 105
Cissoid 225
Cologarithm 80
Columns, difference 73, 119
Components 156
Conchoid 227
Constant 199
derivative of a 264
e 229
of integration 294
π 3, 229
Continuity 254
Coordinates 37, 163, 166
Cosecant 87
Cosine 87
Cosines, Law of 141
Cotangent 87
Coversed sine 87
Cube, duplication of the . . . 226
Cubic equation 10
Curve, area under a 306
distribution 30
exponential 231
frequency 30
logarithmic 230
normal to a 278
normal frequency 31
slope of a 250
Curves, area between two . . . 311
family of 294
trigonometric 110

PAGE

Decimal, repeating 63
De Fermat 166
Degree of an equation . . . 10, 185
Dependence 34
Derivation of formulas 2
Derivative 257
Descartes. 166
Difference, common 57
Difference quotient 174
Differences, columns of . . . 73, 119
Differential 287
Differentiation 258, 293
Diocles 226
Directrix 203, 215, 221
Distance, by integration . . . 313
in trigonometry 85
Division, of a line 171
by logarithms 66
synthetic 319
Duplication of the cube 226

e 229
Eccentricity 215, 221
Elements in a series 57
Elimination in equations . . . 14
Ellipse 211
Equation 9
algebraic 10
cubic 10
degree of an 10, 185
linear 10, 11, 15, 185
locus of an 178, 189
of a locus 178
nonalgebraic 10
numerical 317
quadratic 10, 17, 199
trigonometric 161
Equations, simultaneous . . . 14
theory of 323
Error, percentage 290
Evaluation of a formula . . . 2
Exponent 45
fractional 46
in index notation 50
negative 48
zero as an 50
Exponential curve 231

PAGE

Factorial 53
Factoring 4
special cases of 6
quadratic formula for . . 19, 21
Foci 211, 219
Focus 203, 215, 221
Forces, parallelogram of . . . 155
Formula 1
derivation of a 2
evaluation of a 2
graph of a 28, 40
quadratic 19, 21
Function 34, 243
algebraic 260
derivatives of a 260
logarithmic 121, 361, 374
natural 117, 360, 362
rational integral 317
transcendent 260
trigonometric 83, 86
Function axis 38
Function symbols 34
Functional relations 34

Generation of an angle 83
Geometry, analytic 163
Gradient 103
Graphs, of a formula 28, 40
interpretation of 26, 40, 42
kinds of 25, 27
of logarithms 71
of polynomials 321
solutions by 237
of trigonometric functions . . 110

Haversine 87
Heron 152
Hyperbola 217

Identities, trigonometric . . . 161
Inclination 173
Increment 252
area 306
Induction, mathematical . . . 54
Infinitesimal 254
Integration 293
Intercepts, formula for 184

PAGE

Interpolation, in logarithms . . 72
in function tables 117
Intersection of curves 236

Latus rectum 205, 213
Laws, of Cosines 141
of exponents 46, 48, 50
of logarithms 66, 68, 69
of parallelogram of forces . . 155
of parallelogram of velocities 155
of Sines 136
Length of a line 168
Limit, in series 63
in calculus 254
Line, directed 165
division of a 171
inclination of a 173
length of a 168
midpoint of a 169
projection of a 168
slope of a 174
Lines, parallel 175
perpendicular 175
Loci, miscellaneous 223
straight-line 177
Locus, circle as a 178, 195
of the second degree 203
Logarithmic curve 230
Logarithms 65
to the base 10 65
to the base e 229
hyperbolic 229
natural 229
tables of 357
of trigonometric functions 121, 374

Mantissa 68
Maximum 239, 281
Mean, arithmetic 32, 59
Measures, common 382
Median 32
Mensuration 333
Mercator's projection 39
Midpoint of a line 169
Minimum 239, 281
Modal class (group) 31
Mode 31

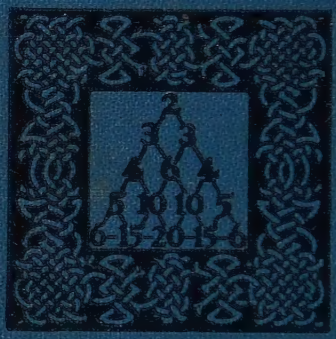
PAGE

Momentum 314
Multiplication by logarithms . . 66

Napier 65
Newton 330
Nicomedes 228
Normal 278
Notation, differential 287
functional 34, 243
index or exponential 50

Ordinates 37, 85, 163
addition of 234
maximum and minimum . . 241
Origin 36, 163

Parabola 203
Parallelogram, of forces 155
of velocities 155
Parameters 294
Period 110
Pi (π) 3, 229
Polynomials 317
Power, of a function . . . 261, 268
root of a 351
Powers, of a binomial 53
by logarithms 66
table of 351
Progression, arithmetic 57
geometric 61
Projection, cabinet 341
of a line 168
Mercator's 39
oblique 341
orthogonal 340
Protractor 90

Quadrant 38, 84, 163
angles in a 84
Quadratic equation 10
factoring 19
formula for 21
locus of . . 195, 203, 212, 218, 223
Quartile 32
Quotient, derivative of a . . . 269
difference 174
of infinitesimals 255

	PAGE
Radian	234
Radicals, derivative of	270
Radius, focal	205, 211, 217
Range	32
Rate, of change	246
instantaneous	250
Ratio, division in a	171
in a series	61
Reduction formulas	112
Relations, functional	34
Remainder Theorem	318
Repeating decimal	63
Resultant	155
Root, logarithm of a	68
Roots, check of	23
of an equation	23, 323
graphic location of	327
table of	351
Secant	87
Sections, conic	203
Series, arithmetic	57
geometric	61
infinite	63
Sign, of coordinates	37, 38, 165
of r	85
of trigonometric functions	88
Simpson	338
Sine	87
Sines, Law of	136
Slope, of a curve	250, 281
of a line	174
Solution, graphic	237
of triangles	123, 133
Speed	248
Square, completing the	17
Successive derivatives	274
Tables, of common measures	382
of logarithmic functions	121, 361, 374
of logarithms	357
of natural functions	100, 117, 362
of powers and roots	351
Tangent, of an angle	87
to a curve	250, 277
Theorem, Binomial	53
Fundamental	323
Remainder	318
Transit	90
Triangle, area of a	129, 151
Heron's formula for	152
oblique	133
right	123
Trigonometry	83
analytic	161
Trisection of an angle	228
Values, maximum and minimum	239
Variables	243
derivatives of	264
limit of	254
Variates	32
Velocities, components of	156
parallelogram of	155
Versed sine	87
Vertex of a parabola	205
Vertices, of an ellipse	213
of a hyperbola	219
Width, focal	205, 213, 219
Work	304, 314
Zero, as an exponent	50
as a limit	254